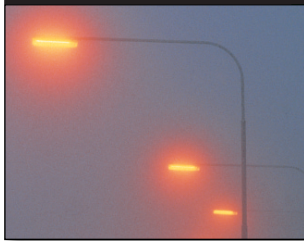


UNIDAD 13



CÁLCULO DE PRIMITIVAS

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Concepto de primitiva

POTENCIAS

1. a) $\int 1 = x$

b) $\int 2 = 2x$

c) $\int \sqrt{2} = \sqrt{2}x$

2. a) $\int 2x = x^2$

b) $\int x = \frac{x^2}{2}$

c) $\int 3x = \frac{3x^2}{2}$

3. a) $\int 7x = \frac{7x^2}{2}$

b) $\int \frac{x}{3} = \frac{x^2}{6}$

c) $\int \sqrt{2}x = \frac{\sqrt{2}x^2}{2}$

4. a) $\int 3x^2 = x^3$

b) $\int x^2 = \frac{x^3}{3}$

c) $\int 2x^2 = \frac{2x^3}{3}$

5. a) $\int 6x^5 = x^6$

b) $\int x^5 = \frac{x^6}{6}$

c) $\int 3x^5 = \frac{3x^6}{6} = \frac{x^6}{2}$

6. a) $\int (-1)x^{-2} = x^{-1} = \frac{1}{x}$

b) $\int x^{-2} = \frac{x^{-1}}{-1} = -\frac{1}{x}$

c) $\int \frac{5}{x^2} = -\frac{5}{x}$

7. a) $\int (k+1)x^k = x^{k+1}$

b) $\int x^k = \frac{x^{k+1}}{k+1}$

8. a) $\int \frac{3}{2}x^{1/2} = x^{3/2} = \sqrt{x^3}$

b) $\int \frac{3}{2}\sqrt{x} = \int \frac{3}{2}x^{1/2} = x^{3/2} = \sqrt{x^3}$

9. a) $\int \sqrt{x} = \frac{2}{3} \int \frac{3}{2}x^{1/2} = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$

b) $\int 7\sqrt{x} = 7 \int \sqrt{x} = \frac{14}{3}\sqrt{x^3}$

10. a) $\int \sqrt{3x} = \int \sqrt{3} \sqrt{x} = \sqrt{3} \int \sqrt{x} = \frac{2\sqrt{3}}{3} \sqrt{x^3} = \frac{2\sqrt{3x^3}}{3}$

b) $\int \frac{\sqrt{2x}}{5} = \int \frac{\sqrt{2}}{5} \sqrt{x} = \frac{\sqrt{2}}{5} \int \sqrt{x} = \frac{\sqrt{2}}{5} \cdot \frac{2}{3} \sqrt{x^3} = \frac{2\sqrt{2}}{15} \sqrt{x^3} = \frac{2\sqrt{2x^3}}{15}$

$$\begin{aligned}
11. \text{ a) } \int \frac{1}{2} x^{-1/2} &= x^{1/2} = \sqrt{x} & \text{ b) } \int \frac{1}{2\sqrt{x}} &= \sqrt{x} \\
12. \text{ a) } \int \frac{3}{2\sqrt{x}} &= 3 \int \frac{1}{2\sqrt{x}} = 3\sqrt{x} & \text{ b) } \int \frac{3}{\sqrt{5x}} &= \frac{6}{5} \int \frac{5}{2\sqrt{5x}} = \frac{6}{5} \sqrt{5x} \\
13. \text{ a) } \int \sqrt{x^3} &= \int x^{3/2} = \frac{x^{5/2}}{5/2} = \frac{2}{5} \sqrt{x^5} & \text{ b) } \int \sqrt{7x^3} &= \sqrt{7} \int \sqrt{x^3} = \frac{2}{5} \sqrt{7x^5} \\
14. \text{ a) } \int \frac{1}{x} &= \ln |x| & \text{ b) } \int \frac{1}{5x} &= \frac{1}{5} \int \frac{5}{5x} = \frac{1}{5} \ln |5x| \\
15. \text{ a) } \int \frac{1}{x+5} &= \ln |x+5| & \text{ b) } \int \frac{3}{2x+6} &= \frac{3}{2} \int \frac{2}{2x+6} = \frac{3}{2} \ln |2x+6| \\
16. \text{ a) } \int \frac{1}{x^3} &= \int x^{-3} = \frac{x^{-2}}{-2} = \frac{-1}{2x^2} & \text{ b) } \int \frac{2}{x^3} &= 2 \int \frac{1}{x^3} = \frac{-2}{2x^2} = \frac{-1}{x^2} \\
17. \text{ a) } \int \frac{1}{(x-3)^3} &= \int (x-3)^{-3} = \frac{(x-3)^{-2}}{-2} = \frac{-1}{2(x-3)^2} \\
& \text{ b) } \int \frac{5}{(x-3)^3} &= 5 \int \frac{1}{(x-3)^3} = \frac{-5}{2(x-3)^2} \\
18. \text{ a) } \int \frac{\sqrt{x}}{\sqrt[3]{x}} &= \int \frac{x^{1/2}}{x^{1/3}} = \int x^{1/6} = \frac{x^{7/6}}{7/6} = \frac{6}{7} \sqrt[6]{x^7} \\
& \text{ b) } \int \frac{\sqrt{3x}}{\sqrt[3]{5x}} &= \int \sqrt[6]{\frac{27x^3}{25x^2}} = \sqrt[6]{\frac{27}{25}} \int x^{1/6} = \frac{6}{7} \sqrt[6]{\frac{27}{25}} \sqrt[6]{x^7} = \frac{6}{7} \sqrt[6]{\frac{27x^7}{25}}
\end{aligned}$$

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TRIGONOMÉTRICAS

$$\begin{aligned}
19. \text{ a) } \int \cos x &= \text{sen } x & \text{ b) } \int 2 \cos x &= 2 \text{ sen } x \\
20. \text{ a) } \int \cos \left(x + \frac{\pi}{2} \right) &= \text{sen} \left(x + \frac{\pi}{2} \right) & \text{ b) } \int \cos 2x &= \frac{1}{2} \int 2 \cos 2x = \frac{1}{2} \text{ sen } 2x \\
21. \text{ a) } \int (-\text{sen } x) &= \cos x & \text{ b) } \int \text{sen } x &= -\cos x \\
22. \text{ a) } \int \text{sen}(x - \pi) &= -\cos(x - \pi) & \text{ b) } \int \text{sen } 2x &= \frac{1}{2} \int 2 \text{ sen } 2x = \frac{-1}{2} \cos 2x
\end{aligned}$$

$$23. \text{ a) } \int 2 \operatorname{sen} x \cos x = \int \operatorname{sen} 2x = \frac{-1}{2} \cos 2x$$

$$\text{b) } \int \operatorname{sen} x \cos x = \frac{1}{2} \int 2 \operatorname{sen} x \cos x = \frac{-1}{4} \cos 2x$$

$$24. \text{ a) } \int (1 + \operatorname{tg}^2 2x) = \frac{1}{2} \int 2(1 + \operatorname{tg}^2 2x) = \frac{1}{2} \operatorname{tg} 2x$$

$$\text{b) } \int \operatorname{tg}^2 2x = \int (1 + \operatorname{tg}^2 2x - 1) = \int (1 + \operatorname{tg}^2 2x) - \int 1 = \frac{1}{2} \operatorname{tg} 2x - x$$

$$25. \text{ a) } \int \frac{1}{1+x^2} = \operatorname{arc} \operatorname{tg} x$$

$$\text{b) } \int \frac{3}{1+x^2} = 3 \operatorname{arc} \operatorname{tg} x$$

$$26. \text{ a) } \int \frac{2}{1+(2x)^2} = \operatorname{arc} \operatorname{tg} (2x)$$

$$\text{b) } \int \frac{1}{1+4x^2} = \frac{1}{2} \int \frac{2}{1+(2x)^2} = \frac{1}{2} \operatorname{arc} \operatorname{tg} (2x)$$

$$27. \text{ a) } \int \frac{1}{\sqrt{1-x^2}} = \operatorname{arc} \operatorname{sen} x$$

$$\text{b) } \int \frac{-1}{\sqrt{1-x^2}} = \operatorname{arc} \operatorname{cos} x$$

EXPONENCIALES

$$28. \text{ a) } \int e^x = e^x$$

$$\text{b) } \int e^{x+1} = e^{x+1}$$

$$29. \text{ a) } \int e^{2x} = \frac{1}{2} \int 2e^{2x} = \frac{1}{2} e^{2x}$$

$$\text{b) } \int e^{2x+1} = \frac{1}{2} \int 2e^{2x+1} = \frac{1}{2} e^{2x+1}$$

$$30. \text{ a) } \int 2x e^{x^2} = e^{x^2}$$

$$\text{b) } \int x e^{x^2} = \frac{1}{2} \int 2x e^{x^2} = \frac{1}{2} e^{x^2}$$

$$31. \text{ a) } \int a^x \ln a = a^x$$

$$\text{b) } \int a^x = \frac{1}{\ln a} \int a^x \ln a = \frac{a^x}{\ln a}$$

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1. Calcula las siguientes integrales:

$$\text{a) } \int 7x^4$$

$$\text{b) } \int \frac{1}{x^2}$$

$$\text{c) } \int \sqrt{x}$$

$$\text{d) } \int \sqrt[3]{5x^2}$$

$$\text{e) } \int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x}$$

$$\text{f) } \int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}$$

$$\begin{aligned}
\text{a)} \int 7x^4 &= 7 \frac{x^5}{5} + k = \frac{7x^5}{5} + k \\
\text{b)} \int \frac{1}{x^2} &= \int x^{-2} = \frac{x^{-1}}{-1} + k = \frac{-1}{x} + k \\
\text{c)} \int \sqrt{x} &= \int x^{1/2} = \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{x^3}}{3} + k \\
\text{d)} \int \sqrt[3]{5x^2} &= \int \sqrt[3]{5} x^{2/3} = \sqrt[3]{5} \frac{x^{5/3}}{5/3} + k = \frac{3\sqrt[3]{5x^5}}{5} + k \\
\text{e)} \int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x} &= \int \frac{x^{1/3}}{3x} + \int \frac{\sqrt{5} x^{3/2}}{3x} = \frac{1}{3} \int x^{-2/3} + \frac{\sqrt{5}}{3} \int x^{1/2} = \\
&= \frac{1}{3} \frac{x^{1/3}}{1/3} + \frac{\sqrt{5}}{3} \frac{x^{3/2}}{3/2} + k = \sqrt[3]{x} + \frac{2\sqrt{5x^3}}{9} + k \\
\text{f)} \int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}} &= \int \frac{\sqrt{5} \cdot x^{3/2}}{\sqrt[3]{3} \cdot x^{1/3}} = \frac{\sqrt{5}}{\sqrt[3]{3}} \int x^{7/6} = \frac{\sqrt{5}}{\sqrt[3]{3}} \frac{x^{13/6}}{13/6} + k = \frac{6\sqrt{5} \sqrt[6]{x^{13}}}{13\sqrt[3]{3}} + k
\end{aligned}$$

2. Calcula:

$$\text{a)} \int \frac{x^4 - 5x^2 + 3x - 4}{x}$$

$$\text{b)} \int \frac{x^4 - 5x^2 + 3x - 4}{x + 1}$$

$$\text{c)} \int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1}$$

$$\text{d)} \int \frac{x^3}{x - 2}$$

$$\text{a)} \int \frac{x^4 - 5x^2 + 3x - 4}{x} = \int \left(x^3 - 5x + 3 - \frac{4}{x} \right) = \frac{x^4}{4} - \frac{5x^2}{2} + 3x - 4 \ln |x| + k$$

$$\begin{aligned}
\text{b)} \int \frac{x^4 - 5x^2 + 3x - 4}{x + 1} &= \int \left(x^3 - x^2 - 4x + 7 - \frac{11}{x + 1} \right) = \\
&= \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 7x - 11 \ln |x + 1| + k
\end{aligned}$$

$$\begin{aligned}
\text{c)} \int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1} &= \int \left(x^2 - 6 + \frac{3x + 2}{x^2 + 1} \right) = \int \left(x^2 - 6 + \frac{3x}{x^2 + 1} + \frac{2}{x^2 + 1} \right) = \\
&= \int x^2 - \int 6 + \frac{3}{2} \int \frac{2x}{x^2 + 1} + 2 \int \frac{1}{x^2 + 1} = \\
&= \frac{x^3}{3} - 6x + \frac{3}{2} \ln(x^2 + 1) + 2 \operatorname{arc} \operatorname{tg} x + k
\end{aligned}$$

$$\text{d)} \int \frac{x^3}{x - 2} = \int \left(x^2 + 2x + 4 + \frac{8}{x - 2} \right) = \frac{x^3}{3} + x^2 + 4x + 8 \ln |x - 2| + k$$

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1. Calcula:

a) $\int \cos^4 x \operatorname{sen} x \, dx$

b) $\int 2^{\operatorname{sen} x} \cos x \, dx$

a) $\int \cos^4 x \operatorname{sen} x \, dx = -\int \cos^4 x (-\operatorname{sen} x) \, dx = -\frac{\cos^5 x}{5} + k$

b) $\int 2^{\operatorname{sen} x} \cos x \, dx = \frac{1}{\ln 2} \int 2^{\operatorname{sen} x} \cos x \cdot \ln 2 \, dx = \frac{2^{\operatorname{sen} x}}{\ln 2} + k$

2. Calcula:

a) $\int \operatorname{cotg} x \, dx$

b) $\int \frac{5x}{x^4 + 1} \, dx$

a) $\int \operatorname{cotg} x \, dx = \int \frac{\cos x}{\operatorname{sen} x} \, dx = \ln |\operatorname{sen} x| + k$

b) $\int \frac{5x}{x^4 + 1} \, dx = \frac{5}{2} \int \frac{2x}{1 + (x^2)^2} \, dx = \frac{5}{2} \operatorname{arc} \operatorname{tg} (x^2) + k$

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3. Calcula: $\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} \, dx$

Hacemos el cambio $x = t^6$, $dx = 6t^5 \, dt$:

$$\begin{aligned} \int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} \, dx &= \int \frac{1}{\sqrt[3]{t^{12}} - \sqrt{t^6}} 6t^5 \, dt = \int \frac{6t^5}{t^4 - t^3} \, dt = \int \frac{6t^2}{t-1} \, dt = 6 \int \frac{t^2}{t-1} \, dt = \\ &= 6 \int \left(t + 1 + \frac{1}{t-1} \right) dt = 6 \int \left(\frac{t^2}{2} + t - \ln |t-1| \right) dt + k = \\ &= 6 \left(\frac{\sqrt[6]{x^2}}{2} + \sqrt[6]{x} - \ln |\sqrt[6]{x} - 1| \right) + k = 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln |\sqrt[6]{x} - 1| + k \end{aligned}$$

4. Calcula: $\int \frac{x}{\sqrt{1-x^2}} \, dx$

Hacemos el cambio $\sqrt{1-x^2} = t \rightarrow 1-x^2 = t^2 \rightarrow x = \sqrt{1-t^2}$

$$dx = \frac{-t}{\sqrt{1-t^2}} \, dt$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{\sqrt{1-t^2}}{t^2} \cdot \frac{-t}{\sqrt{1-t^2}} dt = \int -1 dt = -t + k = -\sqrt{1-x^2} + k$$

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1. Calcula: $\int x \operatorname{sen} x dx$

Llamamos $I = \int x \operatorname{sen} x dx$.

$$\left. \begin{array}{l} u = x, \quad du = dx \\ dv = \operatorname{sen} x dx, \quad v = -\cos x \end{array} \right\} I = -x \cos x + \int \cos x dx = -x \cos x + \operatorname{sen} x + k$$

2. Calcula: $\int x \operatorname{arc} \operatorname{tg} x dx$

Llamamos $I = \int x \operatorname{arc} \operatorname{tg} x dx$.

$$\left. \begin{array}{l} u = \operatorname{arc} \operatorname{tg} x, \quad du = \frac{1}{1+x^2} dx \\ dv = x dx, \quad v = \frac{x^2}{2} \end{array} \right\}$$

$$\begin{aligned} I &= \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \left(\frac{x^2}{1+x^2} \right) dx = \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \\ &= \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} [x - \operatorname{arc} \operatorname{tg} x] + k = \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + k = \\ &= \frac{x^2 + 1}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} x + k \end{aligned}$$

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1. Calcula: $\int \frac{3x^2 - 5x + 1}{x-4} dx$

$$\int \frac{3x^2 - 5x + 1}{x-4} dx = \int \left(3x + 7 + \frac{29}{x-4} \right) dx = \frac{3x^2}{2} + 7x + 29 \ln |x-4| + k$$

2. Calcula: $\int \frac{3x^2 - 5x + 1}{2x+1} dx$

$$\begin{aligned} \int \frac{3x^2 - 5x + 1}{2x+1} dx &= \int \left(\frac{3}{2}x - \frac{13}{4} + \frac{17/4}{2x+1} \right) dx = \\ &= \frac{3}{2} \cdot \frac{x^2}{2} - \frac{13}{4}x - \frac{17}{8} \ln |2x+1| + k = \frac{3x^2}{4} - \frac{13}{4}x - \frac{17}{8} \ln |2x+1| + k \end{aligned}$$

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3. Calcula:

$$\text{a) } \int \frac{5x-3}{x^3-x} dx$$

$$\text{b) } \int \frac{x^2-2x+6}{(x-1)^3} dx$$

a) Descomponemos la fracción:

$$\frac{5x-3}{x^3-x} = \frac{5x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\frac{5x-3}{x^3-x} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$5x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Hallamos A , B y C dando a x los valores 0, 1 y -1:

$$\left. \begin{array}{l} x=0 \Rightarrow -3 = -A \Rightarrow A=3 \\ x=1 \Rightarrow 2 = 2B \Rightarrow B=1 \\ x=-1 \Rightarrow -8 = 2C \Rightarrow C=-4 \end{array} \right\}$$

Así, tenemos que:

$$\int \frac{5x-3}{x^3-x} dx = \int \left(\frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1} \right) dx = 3 \ln|x| + \ln|x-1| - 4 \ln|x+1| + k$$

b) Descomponemos la fracción:

$$\frac{x^2-2x+6}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$x^2-2x+6 = A(x-1)^2 + B(x-1) + C$$

Dando a x los valores 1, 0 y 2, queda:

$$\left. \begin{array}{l} x=1 \Rightarrow 5 = C \\ x=0 \Rightarrow 6 = A - B + C \\ x=2 \Rightarrow 6 = A + B + C \end{array} \right\} \begin{array}{l} A=1 \\ B=0 \\ C=5 \end{array}$$

Por tanto:

$$\int \frac{x^2-2x+6}{(x-1)^3} dx = \int \left(\frac{1}{x-1} + \frac{5}{(x-1)^3} \right) dx = \ln|x-1| - \frac{5}{2(x-1)^2} + k$$

4. Calcula:

$$\text{a) } \int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2} dx$$

$$\text{b) } \int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} dx$$

$$a) x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x-2)(x+2)$$

Descomponemos la fracción:

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2}$$

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} =$$

$$= \frac{Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)}{x^2(x-2)(x+2)}$$

$$x^3 + 22x^2 - 12x + 8 = Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)$$

Hallamos A , B , C y D dando a x los valores 0, 2, -2 y 1:

$$\left. \begin{array}{l} x = 0 \Rightarrow 8 = -4B \Rightarrow B = -2 \\ x = 2 \Rightarrow 80 = 16C \Rightarrow C = 5 \\ x = -2 \Rightarrow 112 = -16D \Rightarrow D = -7 \\ x = 1 \Rightarrow 19 = -3A - 3B + 3C - D \Rightarrow -3A = -9 \Rightarrow A = 3 \end{array} \right\}$$

Por tanto:

$$\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2} dx = \int \left(\frac{3}{x} - \frac{2}{x^2} + \frac{5}{x-2} - \frac{7}{x+2} \right) dx =$$

$$= 3 \ln|x| + \frac{2}{x} + 5 \ln|x-2| - 7 \ln|x+2| + k$$

b) La fracción se puede simplificar:

$$\frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} = \frac{x(x-2)^2}{x(x-2)^2(x+2)} = \frac{1}{x+2}$$

$$\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} dx = \int \frac{1}{x+2} dx = \ln|x+2| + k$$

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EJERCICIOS Y PROBLEMAS PROPUESTOS

PARA PRACTICAR

1 Calcula las siguientes integrales inmediatas:

$$a) \int (4x^2 - 5x + 7) dx \quad b) \int \frac{dx}{\sqrt[5]{x}} \quad c) \int \frac{1}{2x+7} dx \quad d) \int (x - \operatorname{sen} x) dx$$

$$a) \int (4x^2 - 5x + 7) dx = \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + k$$

$$b) \int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5} dx = \frac{x^{4/5}}{4/5} + k = \frac{5\sqrt[5]{x^4}}{4} + k$$

$$c) \int \frac{1}{2x+7} dx = \frac{1}{2} \ln|2x+7| + k$$

$$d) \int (x - \operatorname{sen} x) dx = \frac{x^2}{2} + \operatorname{cox} x + k$$

2 Resuelve estas integrales:

$$a) \int (x^2 + 4x)(x^2 - 1) dx$$

$$b) \int (x-1)^3 dx$$

$$c) \int \sqrt{3x} dx$$

$$d) \int (\operatorname{sen} x + e^x) dx$$

$$a) \int (x^2 + 4x)(x^2 - 1) dx = \int (x^4 + 4x^3 - x^2 - 4x) dx = \frac{x^5}{5} + x^4 - \frac{x^3}{3} - 2x^2 + k$$

$$b) \int (x-1)^3 dx = \frac{(x-1)^4}{4} + k$$

$$c) \int \sqrt{3x} dx = \int \sqrt{3} x^{1/2} dx = \sqrt{3} \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{3}x^{3/2}}{3} + k$$

$$d) \int (\operatorname{sen} x + e^x) dx = -\operatorname{cos} x + e^x + k$$

3 Calcula las integrales siguientes:

$$a) \int \sqrt[3]{\frac{x}{2}} dx$$

$$b) \int \operatorname{sen}(x-4) dx$$

$$c) \int \frac{7}{\operatorname{cos}^2 x} dx$$

$$d) \int (e^x + 3e^{-x}) dx$$

$$a) \int \sqrt[3]{\frac{x}{2}} dx = \frac{1}{\sqrt[3]{2}} \int x^{1/3} dx = \frac{1}{\sqrt[3]{2}} \frac{x^{4/3}}{4/3} + k = \frac{3}{4} \sqrt[3]{\frac{x^4}{2}} + k$$

$$b) \int \operatorname{sen}(x-4) dx = -\operatorname{cos}(x-4) + k$$

$$c) \int \frac{7}{\operatorname{cos}^2 x} dx = 7 \operatorname{tg} x + k$$

$$d) \int (e^x + 3e^{-x}) dx = e^x - 3e^{-x} + k$$

4 Halla estas integrales:

$$a) \int \frac{2}{x} dx$$

$$b) \int \frac{dx}{x-1}$$

$$c) \int \frac{x + \sqrt{x}}{x^2} dx$$

$$d) \int \frac{3}{1+x^2} dx$$

$$a) \int \frac{2}{x} dx = 2 \ln|x| + k$$

$$b) \int \frac{dx}{x-1} = \ln|x-1| + k$$

$$c) \int \frac{x + \sqrt{x}}{x^2} dx = \int \left(\frac{1}{x} + x^{-3/2} \right) dx = \ln|x| - \frac{2}{\sqrt{x}} + k$$

$$d) \int \frac{3}{1+x^2} dx = 3 \operatorname{arc} \operatorname{tg} x + k$$

5 Resuelve las siguientes integrales:

$$a) \int \frac{dx}{x-4} \quad b) \int \frac{dx}{(x-4)^2} \quad c) \int (x-4)^2 dx \quad d) \int \frac{dx}{(x-4)^3}$$

$$a) \int \frac{dx}{x-4} = \ln|x-4| + k$$

$$b) \int \frac{dx}{(x-4)^2} = \frac{-1}{(x-4)} + k$$

$$c) \int (x-4)^2 dx = \frac{(x-4)^3}{3} + k$$

$$d) \int \frac{dx}{(x-4)^3} = \int (x-4)^{-3} dx = \frac{(x-4)^{-2}}{-2} + k = \frac{-1}{2(x-4)^2} + k$$

6 Halla las siguientes integrales del tipo exponencial:

$$a) \int e^{x-4} dx \quad b) \int e^{-2x+9} dx \quad c) \int e^{5x} dx \quad d) \int (3^x - x^3) dx$$

$$a) \int e^{x-4} dx = e^{x-4} + k$$

$$b) \int e^{-2x+9} dx = \frac{-1}{2} \int -2e^{-2x+9} dx = \frac{-1}{2} e^{-2x+9} + k$$

$$c) \int e^{5x} dx = \frac{1}{5} \int 5e^{5x} dx = \frac{1}{5} e^{5x} + k$$

$$d) \int (3^x - x^3) dx = \frac{3^x}{\ln 3} - \frac{x^4}{4} + k$$

7 Resuelve las siguientes integrales del tipo arco tangente:

$$a) \int \frac{dx}{4+x^2} \quad b) \int \frac{4 dx}{3+x^2} \quad c) \int \frac{5 dx}{4x^2+1} \quad d) \int \frac{2 dx}{1+9x^2}$$

$$a) \int \frac{dx}{4+x^2} = \int \frac{1/4}{1+(x/2)^2} dx = \frac{1}{2} \int \frac{1/2}{1+(x/2)^2} dx = \frac{1}{2} \operatorname{arc} \operatorname{tg} \left(\frac{x}{2} \right) + k$$

$$b) \int \frac{4 dx}{3+x^2} = \int \frac{4/3}{1+(x/\sqrt{3})^2} dx = \frac{4\sqrt{3}}{3} \int \frac{1/\sqrt{3}}{1+(x/\sqrt{3})^2} dx = \frac{4\sqrt{3}}{3} \operatorname{arc\,tg} \left(\frac{x}{\sqrt{3}} \right) + k$$

$$c) \int \frac{5 dx}{4x^2+1} = \frac{5}{2} \int \frac{2 dx}{(2x)^2+1} = \frac{5}{2} \operatorname{arc\,tg} (2x) + k$$

$$d) \int \frac{2 dx}{1+9x^2} = \frac{2}{3} \int \frac{3 dx}{1+(3x)^2} = \frac{2}{3} \operatorname{arc\,tg} (3x) + k$$

8 Expresa las siguientes integrales de la forma:

$$\frac{\text{dividendo}}{\text{divisor}} = \text{cociente} + \frac{\text{resto}}{\text{divisor}}$$

y resuélvelas:

$$a) \int \frac{x^2 - 5x + 4}{x+1} dx \quad b) \int \frac{x^2 + 2x + 4}{x+1} dx \quad c) \int \frac{x^3 - 3x^2 + x - 1}{x-2} dx$$

$$a) \int \frac{x^2 - 5x + 4}{x+1} dx = \int \left(x - 6 + \frac{10}{x+1} \right) dx = \frac{x^2}{2} - 6x + 10 \ln|x+1| + k$$

$$b) \int \frac{x^2 + 2x + 4}{x+1} dx = \int \left(x + 1 + \frac{3}{x+1} \right) dx = \frac{x^2}{2} + x + 3 \ln|x+1| + k$$

$$c) \int \frac{x^3 - 3x^2 + x - 1}{x-2} dx = \int \left(x^2 - x - 1 - \frac{3}{x-2} \right) dx =$$

$$= \frac{x^3}{3} - \frac{x^2}{2} - x - 3 \ln|x-2| + k$$

9 Halla estas integrales sabiendo que son del tipo arco seno:

$$a) \int \frac{dx}{\sqrt{1-4x^2}} \quad b) \int \frac{dx}{\sqrt{4-x^2}} \quad c) \int \frac{e^x}{\sqrt{1-e^{2x}}} dx \quad d) \int \frac{dx}{x\sqrt{1-(\ln x)^2}}$$

$$a) \int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2} \int \frac{2 dx}{\sqrt{1-(2x)^2}} = \frac{1}{2} \operatorname{arc\,sen} (2x) + k$$

$$b) \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{1/2 dx}{\sqrt{1-(x/2)^2}} = \operatorname{arc\,sen} \left(\frac{x}{2} \right) + k$$

$$c) \int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}} dx = \operatorname{arc\,sen} (e^x) + k$$

$$d) \int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{1/x dx}{\sqrt{1-(\ln x)^2}} = \operatorname{arc\,sen} (\ln|x|) + k$$

10 Resuelve las integrales siguientes, sabiendo que son de la forma

$$\int f^n(x) \cdot f'(x):$$

a) $\int \cos x \operatorname{sen}^3 x \, dx$ **b)** $\int 2x e^{x^2} \, dx$ **c)** $\int \frac{x \, dx}{(x^2 + 3)^5}$ **d)** $\int \frac{1}{x} \ln^3 x \, dx$

a) $\int \cos x \operatorname{sen}^3 x \, dx = \frac{\operatorname{sen}^4 x}{4} + k$

b) $\int 2x e^{x^2} \, dx = e^{x^2} + k$

c) $\int \frac{x \, dx}{(x^2 + 3)^5} = \frac{1}{2} \int 2x(x^2 + 3)^{-5} \, dx = \frac{1}{2} \frac{(x^2 + 3)^{-4}}{-4} + k = \frac{-1}{8(x^2 + 3)^4} + k$

d) $\int \frac{1}{x} \ln^3 x \, dx = \frac{\ln^4 |x|}{4} + k$

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11 Resuelve las siguientes integrales:

a) $\int x^4 e^{x^5} \, dx$ **b)** $\int x \operatorname{sen} x^2 \, dx$ **c)** $\int \frac{dx}{\sqrt{9 - x^2}}$ **d)** $\int \frac{x \, dx}{\sqrt{x^2 + 5}}$

a) $\int x^4 e^{x^5} \, dx = \frac{1}{5} \int 5x^4 e^{x^5} \, dx = \frac{1}{5} e^{x^5} + k$

b) $\int x \operatorname{sen} x^2 \, dx = \frac{1}{2} \int 2x \operatorname{sen} x^2 \, dx = \frac{-1}{2} \cos x^2 + k$

c) $\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{1/3 \, dx}{\sqrt{1 - (x/3)^2}} = \operatorname{arc} \operatorname{sen} \left(\frac{x}{3} \right) + k$

d) $\int \frac{x \, dx}{\sqrt{x^2 + 5}} = \sqrt{x^2 + 5} + k$

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12 Resuelve las siguientes integrales:

a) $\int \operatorname{sen} x \cos x \, dx$ **b)** $\int \frac{\operatorname{sen} x \, dx}{\cos^5 x}$ **c)** $\int \sqrt{(x + 3)^5} \, dx$ **d)** $\int \frac{-3x}{2 - 6x^2} \, dx$

a) $\int \operatorname{sen} x \cos x \, dx = \frac{\operatorname{sen}^2 x}{2} + k$

$$b) \int \frac{\operatorname{sen} x \, dx}{\cos^5 x} = -\int (-\operatorname{sen} x) \cdot \cos^{-5} x \, dx = \frac{-\cos^{-4} x}{-4} + k = \frac{1}{4 \cos^4 x} + k$$

$$c) \int \sqrt{(x+3)^5} \, dx = \int (x+3)^{5/2} \, dx = \frac{(x+3)^{7/2}}{7/2} + k = \frac{2\sqrt{(x+3)^7}}{7} + k$$

$$d) \int \frac{-3x}{2-6x^2} \, dx = \frac{1}{4} \int \frac{-12x}{2-6x^2} \, dx = \frac{1}{4} \ln|2-6x^2| + k$$

13 Resuelve las siguientes integrales:

$$a) \int \sqrt{x^2-2x} (x-1) \, dx \qquad b) \int \operatorname{tg} x \sec^2 x \, dx$$

$$c) \int \frac{(1+\ln x)^2}{x} \, dx \qquad d) \int \sqrt{(1+\cos x)^3} \operatorname{sen} x \, dx$$

$$a) \int \sqrt{x^2-2x} (x-1) \, dx = \frac{1}{2} \int \sqrt{x^2-2x} (2x-2) \, dx = \frac{1}{2} \int (x^2-2x)^{1/2} (2x-2) \, dx =$$

$$= \frac{1}{2} \frac{(x^2-2x)^{3/2}}{3/2} + k = \frac{\sqrt{(x^2-2x)^3}}{3} + k$$

$$b) \int \operatorname{tg} x \sec^2 x \, dx = \frac{\operatorname{tg}^2 x}{2} + k$$

$$c) \int \frac{(1+\ln x)^2}{x} \, dx = \int (1+\ln x)^2 \cdot \frac{1}{x} \, dx = \frac{(1+\ln|x|)^3}{3} + k$$

$$d) \int \sqrt{(1+\cos x)^3} \operatorname{sen} x \, dx = -\int (1+\cos x)^{3/2} (-\operatorname{sen} x) \, dx = -\frac{(1+\cos x)^{5/2}}{5/2} + k =$$

$$= \frac{-2\sqrt{(1+\cos x)^5}}{5} + k$$

14 Aplica la integración por partes para resolver las siguientes integrales:

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$$a) \int x \ln x \, dx \qquad b) \int e^x \cos x \, dx \qquad c) \int x^2 \operatorname{sen} x \, dx \qquad d) \int x^2 e^{2x} \, dx$$

$$e) \int \cos(\ln x) \, dx \qquad f) \int x^2 \ln x \, dx \qquad g) \int \operatorname{arc} \operatorname{tg} x \, dx \qquad h) \int (x+1)^2 e^x \, dx$$

$$a) \int x \ln x \, dx$$

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} \, dx \\ dv = x \, dx \rightarrow v = \frac{x^2}{2} \end{cases}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + k$$

$$b) \int e^x \cos x \, dx$$

$$\begin{cases} u = e^x \rightarrow du = e^x \, dx \\ dv = \cos x \, dx \rightarrow v = \text{sen } x \end{cases}$$

$$\int e^x \cos x \, dx = e^x \text{sen } x - \underbrace{\int e^x \text{sen } x \, dx}_{I_1}$$

$$\begin{cases} u_1 = e^x \rightarrow du_1 = e^x \, dx \\ dv_1 = \text{sen } x \, dx \rightarrow v_1 = -\cos x \end{cases}$$

$$I_1 = -e^x \cos x + \int e^x \cos x \, dx$$

Por tanto:

$$\int e^x \cos x \, dx = e^x \text{sen } x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \text{sen } x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{e^x \text{sen } x + e^x \cos x}{2} + k$$

$$c) \int x^2 \text{sen } x \, dx$$

$$\begin{cases} u = x^2 \rightarrow du = 2x \, dx \\ dv = \text{sen } x \, dx \rightarrow v = -\cos x \end{cases}$$

$$\int x^2 \text{sen } x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2 \underbrace{\int x \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x \rightarrow du_1 = dx \\ dv_1 = \cos x \, dx \rightarrow v_1 = \text{sen } x \end{cases}$$

$$I_1 = x \text{sen } x - \int \text{sen } x \, dx = x \text{sen } x + \cos x$$

Por tanto:

$$\int x^2 \text{sen } x \, dx = -x^2 \cos x + 2x \text{sen } x + 2 \cos x + k$$

$$d) \int x^2 e^{2x} \, dx$$

$$\begin{cases} u = x^2 \rightarrow du = 2x \, dx \\ dv = e^{2x} \, dx \rightarrow v = \frac{1}{2} e^{2x} \end{cases}$$

$$\int x^2 e^{2x} dx = \frac{x^2}{2} e^{2x} - \underbrace{\int x e^{2x} dx}_{I_1}$$

$$\begin{cases} u_1 = x \rightarrow du_1 = dx \\ dv_1 = e^{2x} dx \rightarrow v_1 = \frac{1}{2} e^{2x} \end{cases}$$

$$I_1 = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x}$$

Por tanto: $\int x^2 e^{2x} dx = \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + k = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) e^{2x} + k$

e) $\int \cos(\ln x) dx$

$$\begin{cases} u = \cos(\ln x) \rightarrow du = -\operatorname{sen}(\ln x) \cdot \frac{1}{x} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \underbrace{\int \operatorname{sen}(\ln x) dx}_{I_1}$$

$$\begin{cases} u_1 = \operatorname{sen}(\ln x) \rightarrow du_1 = \cos(\ln x) \cdot \frac{1}{x} dx \\ dv_1 = dx \rightarrow v_1 = x \end{cases}$$

$$I_1 = x \operatorname{sen}(\ln x) - \int \cos(\ln x) dx$$

Por tanto:

$$\int \cos(\ln x) dx = x \cos(\ln x) + x \operatorname{sen}(\ln x) - \int \cos(\ln x) dx$$

$$2 \int \cos(\ln x) dx = x \cos(\ln x) + x \operatorname{sen}(\ln x)$$

$$\int \cos(\ln x) dx = \frac{x \cos(\ln x) + x \operatorname{sen}(\ln x)}{2} + k$$

f) $\int x^2 \ln x dx$

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = x^2 dx \rightarrow v = \frac{x^3}{3} \end{cases}$$

$$\int x^2 \ln x dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + k$$

$$g) \int \operatorname{arc} \operatorname{tg} x \, dx$$

$$\begin{cases} u = \operatorname{arc} \operatorname{tg} x & \rightarrow & du = \frac{1}{1+x^2} dx \\ dv = dx & \rightarrow & v = x \end{cases}$$

$$\begin{aligned} \int \operatorname{arc} \operatorname{tg} x &= x \operatorname{arc} \operatorname{tg} x - \int \frac{1}{1+x^2} dx = x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = \\ &= x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \ln(1+x^2) + k \end{aligned}$$

$$h) \int (x+1)^2 e^x \, dx$$

$$\begin{cases} u = (x+1)^2 & \rightarrow & du = 2(x+1) dx \\ dv = e^x dx & \rightarrow & v = e^x \end{cases}$$

$$\int (x+1)^2 e^x \, dx = (x+1)^2 e^x - 2 \underbrace{\int (x+1) e^x \, dx}_{I_1}$$

$$\begin{cases} u_1 = (x+1) & \rightarrow & du_1 = dx \\ dv_1 = e^x dx & \rightarrow & v_1 = e^x \end{cases}$$

$$I_1 = (x+1) e^x - \int e^x dx = (x+1) e^x - e^x = (x+1-1) e^x = x e^x$$

Por tanto:

$$\begin{aligned} \int (x+1)^2 e^x \, dx &= (x+1)^2 e^x - 2x e^x + k = \\ &= (x^2 + 2x + 1 - 2x) e^x + k = (x^2 + 1) e^x + k \end{aligned}$$

15 Calcula $\int \cos^4 x \, dx$ utilizando la expresión: $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$

$$\cos^4 x = \left(\frac{1}{2} + \frac{\cos 2x}{2} \right)^2 = \frac{1}{4} + \frac{\cos^2 2x}{4} + \frac{\cos 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{4} \left(\frac{1}{2} + \frac{\cos 4x}{2} \right) + \frac{\cos 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2} = \frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}$$

Por tanto:

$$\int \cos^4 x \, dx = \int \left(\frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2} \right) dx = \frac{3}{8} x + \frac{\operatorname{sen} 4x}{32} + \frac{\operatorname{sen} 2x}{2} + k$$

16 Determina el valor de las integrales propuestas en los ejercicios siguientes utilizando la fórmula de integración por partes:

a) $\int x^2 e^{3x} dx$ b) $\int \frac{x}{e^x} dx$ c) $\int 3x \cos x dx$ d) $\int x^3 \operatorname{sen} x dx$

a) $\int x^2 e^{3x} dx$

$$\begin{cases} u = x^2 \rightarrow du = 2x dx \\ dv = e^{3x} dx \rightarrow v = \frac{1}{3} e^{3x} \end{cases}$$

$$\int x^2 e^{3x} dx = \frac{x^2}{3} e^{3x} - \frac{2}{3} \underbrace{\int x e^{3x} dx}_{I_1}$$

$$\begin{cases} u_1 = x \rightarrow du_1 = dx \\ dv_1 = e^{3x} dx \rightarrow v_1 = \frac{1}{3} e^{3x} \end{cases}$$

$$I_1 = \frac{x}{3} e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x}$$

Por tanto:

$$\int x^2 e^{3x} dx = \frac{x^2}{3} e^{3x} - \frac{2x}{9} e^{3x} + \frac{2}{27} e^{3x} + k = \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) e^{3x} + k$$

b) $\int \frac{x}{e^x} dx = \int x e^{-x} dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = e^{-x} dx \rightarrow v = -e^{-x} \end{cases}$$

$$\int \frac{x}{e^x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + k = \frac{-x}{e^x} - \frac{1}{e^x} + k = \frac{-x-1}{e^x} + k$$

c) $\int 3x \cos x dx$

$$\begin{cases} u = 3x \rightarrow du = 3 dx \\ dv = \cos x dx \rightarrow v = \operatorname{sen} x \end{cases}$$

$$\int 3x \cos x dx = 3x \operatorname{sen} x - 3 \int \operatorname{sen} x dx = 3x \operatorname{sen} x + 3 \cos x + k$$

d) $\int x^3 \operatorname{sen} x dx$

$$\begin{cases} u = x^3 \rightarrow du = 3x^2 dx \\ dv = \operatorname{sen} x dx \rightarrow v = -\cos x \end{cases}$$

$$\int x^3 \operatorname{sen} x \, dx = -x^3 \cos x + 3 \underbrace{\int x^2 \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x^2 \rightarrow du_1 = 2x \, dx \\ dv_1 = \cos x \, dx \rightarrow v_1 = \operatorname{sen} x \end{cases}$$

$$I_1 = x^2 \operatorname{sen} x - 2 \underbrace{\int x \operatorname{sen} x \, dx}_{I_2}$$

$$\begin{cases} u_2 = x \rightarrow du_2 = dx \\ dv_2 = \operatorname{sen} x \, dx \rightarrow v_2 = -\cos x \end{cases}$$

$$I_2 = -x \cos x + \int \cos x \, dx = -x \cos x + \operatorname{sen} x$$

Así: $I_1 = x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x$

Por tanto:

$$\int x^3 \operatorname{sen} x \, dx = -x^3 \cos x + 3x^2 \operatorname{sen} x + 6x \cos x - 6 \operatorname{sen} x + k$$

17 Determina el valor de las integrales que se proponen a continuación:

a) $\int x \cdot 2^{-x} \, dx$ **b)** $\int \operatorname{arc} \cos x \, dx$ **c)** $\int x \cos 3x \, dx$ **d)** $\int x^5 e^{-x^3} \, dx$

a) $\int x \cdot 2^{-x} \, dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = 2^{-x} \, dx \rightarrow v = \frac{-2^{-x}}{\ln 2} \end{cases}$$

$$\begin{aligned} \int x 2^{-x} \, dx &= \frac{-x \cdot 2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2} \, dx = \frac{-x \cdot 2^{-x}}{\ln 2} + \frac{1}{\ln 2} \int 2^{-x} \, dx = \\ &= \frac{-x \cdot 2^{-x}}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2} + k \end{aligned}$$

b) $\int \operatorname{arc} \cos x \, dx$

$$\begin{cases} u = \operatorname{arc} \cos x \rightarrow du = \frac{-1}{\sqrt{1-x^2}} \, dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int \operatorname{arc} \cos x \, dx = x \operatorname{arc} \cos x - \int \frac{-x}{\sqrt{1-x^2}} \, dx = x \operatorname{arc} \cos x - \sqrt{1-x^2} + k$$

$$c) \int x \cos 3x \, dx$$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = \cos 3x \, dx \rightarrow v = \frac{1}{3} \operatorname{sen} 3x \end{cases}$$

$$\int x \cos 3x \, dx = \frac{x}{3} \operatorname{sen} 3x - \frac{1}{3} \int \operatorname{sen} 3x \, dx = \frac{x}{3} \operatorname{sen} 3x + \frac{1}{9} \cos 3x + k$$

$$d) \int x^5 e^{-x^3} \, dx = \int \underbrace{x^3}_u \cdot \underbrace{x^2 e^{-x^3} \, dx}_{dv}$$

$$\begin{cases} u = x^3 \rightarrow du = 3x^2 \, dx \\ dv = x^2 e^{-x^3} \, dx \rightarrow v = \frac{-1}{3} e^{-x^3} \end{cases}$$

$$\begin{aligned} \int x^5 e^{-x^3} \, dx &= \frac{-x^3}{3} e^{-x^3} + \int x^2 e^{-x^3} \, dx = \frac{-x^3}{3} e^{-x^3} - \frac{1}{3} e^{-x^3} + k = \\ &= \frac{(-x^3 - 1)}{3} e^{-x^3} + k \end{aligned}$$

18 En el ejercicio resuelto 7 a), se ha calculado la integral $\int \operatorname{sen}^2 x \, dx$ aplicando la igualdad:

$$\operatorname{sen}^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$$

Vamos a obtenerla, ahora, mediante la integración por partes, haciendo:

$$\begin{cases} u = \operatorname{sen} x \rightarrow du = \cos x \, dx \\ dv = \operatorname{sen} x \, dx \rightarrow v = -\cos x \end{cases}$$

$$\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + \int \cos^2 x \, dx$$

Si con esta nueva integral procedemos como con la anterior, llegaríamos a una identidad inútil (“se nos va todo”). Compruébalo.

Sin embargo, si hacemos $\cos^2 x = 1 - \operatorname{sen}^2 x$, se resuelve con facilidad. Termina la integral.

• Si aplicáramos el método de integración por partes a la integral $\int \cos^2 x \, dx$, tendríamos que:

$$\begin{cases} u = \cos x \rightarrow du = -\operatorname{sen} x \, dx \\ dv = \cos x \, dx \rightarrow v = \operatorname{sen} x \end{cases}$$

$$\text{Por tanto, quedaría: } \int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + \operatorname{sen} x \cos x + \int \operatorname{sen}^2 x \, dx$$

En efecto, es una identidad inútil (“se nos va todo”).

- Sin embargo, si hacemos $\cos^2 x = 1 - \operatorname{sen}^2 x$, tenemos que:

$$\begin{aligned}\int \operatorname{sen}^2 x \, dx &= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx = \\ &= -\operatorname{sen} x \cos x + \int dx - \int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x \, dx\end{aligned}$$

Por tanto:

$$\begin{aligned}2 \int \operatorname{sen}^2 x \, dx &= -\operatorname{sen} x \cos x + x \\ \int \operatorname{sen}^2 x \, dx &= \frac{-\operatorname{sen} x \cos x + x}{2} + k = \frac{1}{2}x - \frac{1}{4} \operatorname{sen} 2x + k\end{aligned}$$

19 Determina el valor de las integrales racionales propuestas en los siguientes ejercicios:

$$\begin{array}{ll} \text{a) } \int \frac{x+2}{x^2+1} \, dx & \text{b) } \int \frac{1}{(x^2-1)^2} \, dx \\ \text{c) } \int \frac{2x^2+7x-1}{x^3+x^2-x-1} \, dx & \text{d) } \int \frac{2x^2+5x-1}{x^3+x^2-2x} \, dx \end{array}$$

$$\text{a) } \int \frac{x+2}{x^2+1} \, dx = \frac{1}{2} \int \frac{2x}{x^2+1} \, dx + \int \frac{2}{x^2+1} \, dx = \frac{1}{2} \ln(x^2+1) + 2 \operatorname{arc} \operatorname{tg} x + k$$

$$\text{b) } \int \frac{1}{(x^2-1)^2} \, dx = \frac{1}{(x-1)^2(x+1)^2} \, dx$$

Descomponemos en fracciones simples:

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2}{(x-1)^2(x+1)^2}$$

$$1 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2$$

Calculamos A , B , C y D , dando a x los valores 1, -1, 0 y 2:

$$\left. \begin{array}{l} x=1 \rightarrow 1=4B \rightarrow B=1/4 \\ x=-1 \rightarrow 1=4D \rightarrow D=1/4 \\ x=0 \rightarrow 1=-A+B+C+D \rightarrow 1/2=-A+C \\ x=2 \rightarrow 1=9A+9B+3C+D \rightarrow -3/2=9A+3C \rightarrow -1/2=3A+C \end{array} \right\} \begin{array}{l} A=-1/4 \\ B=1/4 \\ C=1/4 \\ D=1/4 \end{array}$$

$$\begin{aligned}\int \frac{1}{(x^2-1)^2} \, dx &= \int \frac{-1/4}{x-1} \, dx + \int \frac{1/4}{(x-1)^2} \, dx + \int \frac{1/4}{x+1} \, dx + \int \frac{1/4}{(x+1)^2} \, dx = \\ &= \frac{-1}{4} \ln|x-1| - \frac{1}{4} \cdot \frac{1}{(x-1)} + \frac{1}{4} \ln|x+1| - \frac{1}{4} \cdot \frac{1}{(x+1)} + k =\end{aligned}$$

$$= \frac{-1}{4} \left[\ln|x-1| + \frac{1}{x-1} - \ln|x+1| + \frac{1}{x+1} \right] + k =$$

$$= \frac{-1}{4} \left[\ln \left| \frac{x-1}{x+1} \right| + \frac{2x}{x^2-1} \right] + k$$

$$c) \int \frac{2x^2 + 7x - 1}{x^3 + x^2 - x - 1} dx = \int \frac{2x^2 + 7x - 1}{(x-1)(x+1)^2} dx$$

Descomponemos en fracciones simples:

$$\frac{2x^2 + 7x - 1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+1^2}$$

$$\frac{2x^2 + 7x - 1}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$2x^2 + 7x - 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 1 \rightarrow 8 = 4A \rightarrow A = 2 \\ x = -1 \rightarrow -6 = -2C \rightarrow C = 3 \\ x = 0 \rightarrow -1 = A - B - C \rightarrow B = 0 \end{array} \right\}$$

Por tanto:

$$\int \frac{2x^2 + 7x - 1}{x^3 + x^2 - x - 1} dx = \int \frac{2}{x-1} dx + \int \frac{3}{(x+1)^2} dx = 2 \ln|x-1| - \frac{3}{x+1} + k$$

$$d) \int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx = \int \frac{2x^2 + 5x - 1}{x(x-1)(x+2)} dx$$

Descomponemos en fracciones simples:

$$\frac{2x^2 + 5x - 1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\frac{2x^2 + 5x - 1}{x(x-1)(x+2)} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

$$2x^2 + 5x - 1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 0 \rightarrow -1 = -2A \rightarrow A = 1/2 \\ x = 1 \rightarrow 6 = 3B \rightarrow B = 2 \\ x = -2 \rightarrow -3 = 6C \rightarrow C = -1/2 \end{array} \right\}$$

Por tanto:

$$\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx = \int \frac{1/2}{x} dx + \int \frac{2}{x-1} dx + \int \frac{-1/2}{x+2} dx =$$

$$= \frac{1}{2} \ln|x| + 2 \ln|x-1| - \frac{1}{2} \ln|x+2| + k = \ln \left(\frac{(x-1)^2 \sqrt{x}}{\sqrt{x+2}} \right) + k$$

20 Resuelve las siguientes integrales:

a) $\int \frac{2x-4}{(x-1)^2(x+3)} dx$ b) $\int \frac{2x+3}{(x-2)(x+5)} dx$

c) $\int \frac{1}{(x-1)(x+3)^2} dx$ d) $\int \frac{3x-2}{x^2-4} dx$

a) $\int \frac{2x-4}{(x-1)^2(x+3)} dx$

Descomponemos en fracciones simples:

$$\frac{2x-4}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$\frac{2x-4}{(x-1)^2(x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$2x-4 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x=1 \rightarrow -2 = 4B \rightarrow B = -1/2 \\ x=-3 \rightarrow -10 = 16C \rightarrow C = -5/8 \\ x=0 \rightarrow -4 = -3A + 3B + C \rightarrow A = 5/8 \end{array} \right\}$$

Por tanto:

$$\int \frac{2x-4}{(x-1)^2(x+3)} dx = \int \frac{5/8}{x-1} dx + \int \frac{-1/2}{(x-1)^2} dx + \int \frac{-5/8}{x+3} dx =$$

$$= \frac{5}{8} \ln|x-1| + \frac{1}{2} \cdot \frac{1}{(x-1)} - \frac{5}{8} \ln|x+3| + k = \frac{5}{8} \ln \left| \frac{x-1}{x+3} \right| + \frac{1}{2x-2} + k$$

b) $\int \frac{2x+3}{(x-2)(x+5)} dx$

Descomponemos en fracciones simples:

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5) + B(x-2)}{(x-2)(x+5)}$$

$$2x+3 = A(x+5) + B(x-2)$$

Hallamos A y B :

$$\left. \begin{array}{l} x = 2 \rightarrow 7 = 7A \rightarrow A = 1 \\ x = -5 \rightarrow -7 = -7B \rightarrow B = 1 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2x+3}{(x-2)(x+5)} dx &= \int \frac{1}{x-2} dx + \int \frac{1}{x+5} dx = \\ &= \ln|x-2| + \ln|x+5| + k = \ln|(x-2)(x+5)| + k \end{aligned}$$

$$c) \int \frac{1}{(x-1)(x+3)^2} dx$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{1}{(x-1)(x+3)^2} &= \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2} \\ \frac{1}{(x-1)(x+3)^2} &= \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)(x+3)^2} \\ 1 &= A(x+3)^2 + B(x-1)(x+3) + C(x-1) \end{aligned}$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 1 \rightarrow 1 = 16A \rightarrow A = 1/16 \\ x = -3 \rightarrow 1 = -4C \rightarrow C = -1/4 \\ x = 0 \rightarrow 1 = 9A - 3B - C \rightarrow B = -1/16 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{1}{(x-1)(x+3)^2} dx &= \int \frac{1/16}{x-1} dx + \int \frac{-1/16}{x+3} dx + \int \frac{-1/4}{(x+3)^2} dx = \\ &= \frac{1}{16} \ln|x-1| - \frac{1}{16} \ln|x+3| + \frac{1}{4} \cdot \frac{1}{(x+3)} + k = \\ &= \frac{1}{16} \ln \left| \frac{x-1}{x+3} \right| + \frac{1}{4(x+3)} + k \end{aligned}$$

$$d) \int \frac{3x-2}{x^2-4} dx = \int \frac{3x-2}{(x-2)(x+2)} dx$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{3x-2}{(x-2)(x+2)} &= \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)} \\ 3x-2 &= A(x+2) + B(x-2) \end{aligned}$$

Hallamos A y B :

$$\left. \begin{array}{l} x = 2 \rightarrow 4 = 4A \rightarrow A = 1 \\ x = -2 \rightarrow -8 = -4B \rightarrow B = 2 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{3x-2}{x^2-4} dx &= \int \frac{1}{x-2} dx + \int \frac{2}{x+2} dx = \\ &= \ln|x-2| + 2 \ln|x+2| + k = \ln[|x-2|(x+2)^2] + k \end{aligned}$$

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21 **S** **Calcula:**

a) $\int \frac{dx}{x^2-x-2}$

b) $\int \frac{x^4+2x-6}{x^3+x^2-2x} dx$

c) $\int \frac{5x^2}{x^3-3x^2+3x-1} dx$

d) $\int \frac{2x-3}{x^3-2x^2-9x+18} dx$

a) $\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x+1)(x-2)}$

Descomponemos en fracciones simples:

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

$$1 = A(x-2) + B(x+1)$$

Hallamos A y B :

$$\left. \begin{array}{l} x = -1 \rightarrow 1 = -3A \rightarrow A = -1/3 \\ x = 2 \rightarrow 1 = 3B \rightarrow B = 1/3 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{dx}{x^2-x-2} dx &= \int \frac{-1/3}{x+1} dx + \int \frac{1/3}{x-2} dx = \\ &= \frac{-1}{3} \ln|x+1| + \frac{1}{3} \ln|x-2| + k = \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + k \end{aligned}$$

b) $\int \frac{x^4+2x-6}{x^3+x^2-2x} dx = \int \left(x-1 + \frac{3x^2-6}{x(x-1)(x+2)} \right) dx$

Descomponemos en fracciones simples:

$$\frac{3x^2-6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\frac{3x^2 - 6}{x(x-1)(x+2)} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

$$3x^2 - 6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 0 \rightarrow -6 = -2A \rightarrow A = 3 \\ x = 1 \rightarrow -3 = 3B \rightarrow B = -1 \\ x = -2 \rightarrow 6 = 6C \rightarrow C = 1 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{x^4 + 2x - 6}{x^3 + x^2 - 2x} dx &= \int \left(x - 1 + \frac{3}{x} - \frac{1}{x-1} + \frac{1}{x+2} \right) dx = \\ &= \frac{x^2}{2} - x + 3 \ln|x| - \ln|x-1| + \ln|x+2| + k = \\ &= \frac{x^2}{2} - x + \ln \left| \frac{x^3(x+2)}{x-1} \right| + k \end{aligned}$$

$$c) \int \frac{5x^2}{x^3 - 3x^2 + 3x - 1} dx = \int \frac{5x^2}{(x-1)^3} dx$$

Descomponemos en fracciones simples:

$$\frac{5x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$5x^2 = A(x-1)^2 + B(x-1) + C$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 1 \rightarrow 5 = C \\ x = 2 \rightarrow 20 = A + B + C \\ x = 0 \rightarrow 0 = A - B + C \end{array} \right\} \begin{array}{l} A = 5 \\ B = 10 \\ C = 5 \end{array}$$

Por tanto:

$$\begin{aligned} \int \frac{5x^2}{x^3 - 3x^2 + 3x - 1} dx &= \int \left(\frac{5}{x-1} + \frac{10}{(x-1)^2} + \frac{5}{(x-1)^3} \right) dx = \\ &= 5 \ln|x-1| - \frac{10}{x-1} - \frac{5}{2(x-1)^2} + k \end{aligned}$$

$$d) \int \frac{2x-3}{x^3 - 2x^2 - 9x + 18} dx = \int \frac{2x-3}{(x-2)(x-3)(x+3)} dx$$

Descomponemos en fracciones simples:

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+3}$$

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)}{(x-2)(x-3)(x+3)}$$

$$2x-3 = A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x=2 \rightarrow 1 = -5A \rightarrow A = -1/5 \\ x=3 \rightarrow 3 = 6B \rightarrow B = 1/2 \\ x=-3 \rightarrow -9 = 30C \rightarrow C = -3/10 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2x-3}{x^3-2x^2-9x+18} dx &= \int \left(\frac{-1/5}{x-2} + \frac{1/2}{x-3} + \frac{-3/10}{x+3} \right) dx = \\ &= \frac{-1}{5} \ln|x-2| + \frac{1}{2} \ln|x-3| - \frac{3}{10} \ln|x+3| + k \end{aligned}$$

22 Resuelve las integrales:

$$\text{a) } \int \frac{\ln x}{x} dx \qquad \text{b) } \int \frac{1 - \operatorname{sen} x}{x + \cos x} dx \qquad \text{c) } \int \frac{1}{x \ln x} dx$$

$$\text{d) } \int \frac{1 + e^x}{e^x + x} dx \qquad \text{e) } \int \frac{\operatorname{sen}(1/x)}{x^2} dx \qquad \text{f) } \int \frac{2x-3}{x+2} dx$$

$$\text{g) } \int \frac{\operatorname{arc} \operatorname{tg} x}{1+x^2} dx \qquad \text{h) } \int \frac{\operatorname{sen} x}{\cos^4 x} dx$$

$$\text{a) } \int \frac{\ln x}{x} dx = \int \frac{1}{x} \ln x dx = \frac{\ln^2|x|}{2} + k$$

$$\text{b) } \int \frac{1 - \operatorname{sen} x}{x + \cos x} dx = \ln|x + \cos x| + k$$

$$\text{c) } \int \frac{1}{x \ln x} dx = \int \frac{1/x}{\ln x} dx = \ln|\ln|x|| + k$$

$$\text{d) } \int \frac{1 + e^x}{e^x + x} dx = \ln|e^x + x| + k$$

$$\text{e) } \int \frac{\operatorname{sen}(1/x)}{x^2} dx = - \int \frac{-1}{x^2} \operatorname{sen}\left(\frac{1}{x}\right) dx = \cos\left(\frac{1}{x}\right) + k$$

$$\text{f) } \int \frac{2x-3}{x+2} dx = \int \left(2 - \frac{7}{x+2} \right) dx = 2x - 7 \ln|x+2| + k$$

$$g) \int \frac{\operatorname{arc\,tg} x}{1+x^2} dx = \int \frac{1}{1+x^2} \operatorname{arc\,tg} x dx = \frac{\operatorname{arc\,tg}^2 x}{2} + k$$

$$h) \int \frac{\operatorname{sen} x}{\cos^4 x} dx = - \int (-\operatorname{sen} x)(\cos x)^{-4} dx = \frac{-(\cos x)^{-3}}{-3} + k = \frac{1}{3 \cos^3 x} + k$$

23 Calcula las integrales indefinidas:

$$a) \int \frac{\operatorname{sen} \sqrt{x}}{\sqrt{x}} dx$$

$$b) \int \ln(x-3) dx$$

$$c) \int \frac{\ln \sqrt{x}}{\sqrt{x}} dx$$

$$d) \int \ln(x^2+1) dx$$

$$e) \int (\ln x)^2 dx$$

$$f) \int e^x \cos e^x dx$$

$$g) \int \frac{1}{1-x^2} dx$$

$$h) \int \frac{(1-x)^2}{1+x} dx$$

$$a) \int \frac{\operatorname{sen} \sqrt{x}}{\sqrt{x}} dx = -2 \int \frac{1}{2\sqrt{x}} (-\operatorname{sen} \sqrt{x}) dx = -2 \cos(\sqrt{x}) + k$$

$$b) \int \ln(x-3) dx$$

$$\begin{cases} u = \ln(x-3) \rightarrow du = \frac{1}{x-3} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\begin{aligned} \int \ln(x-3) dx &= x \ln|x-3| - \int \frac{x}{x-3} dx = x \ln|x-3| - \int \left(1 + \frac{3}{x-3}\right) dx = \\ &= x \ln|x-3| - x - 3 \ln|x-3| + k = (x-3) \ln|x-3| - x + k \end{aligned}$$

$$c) \int \frac{\ln \sqrt{x}}{\sqrt{x}} dx$$

$$\begin{cases} u = \ln \sqrt{x} \rightarrow du = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x} dx \\ v = \frac{1}{\sqrt{x}} dx \rightarrow dv = -\frac{1}{2\sqrt{x}} dx \end{cases}$$

$$\begin{aligned} \int \frac{\ln \sqrt{x}}{\sqrt{x}} dx &= 2\sqrt{x} \ln \sqrt{x} - \int \frac{2\sqrt{x}}{2x} dx = 2\sqrt{x} \ln \sqrt{x} - \int \frac{1}{\sqrt{x}} dx = \\ &= 2\sqrt{x} \ln \sqrt{x} - 2\sqrt{x} + k = 2\sqrt{x} (\ln \sqrt{x} - 1) + k \end{aligned}$$

$$d) \int \ln(x^2 + 1) dx$$

$$\begin{cases} u = \ln(x^2 + 1) \rightarrow du = \frac{2x}{x^2 + 1} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\begin{aligned} \int \ln(x^2 + 1) dx &= x \ln(x^2 + 1) - \int \frac{2x^2}{x^2 + 1} dx = \\ &= x \ln(x^2 + 1) - \int \left(2 - \frac{2}{x^2 + 1}\right) dx = x \ln(x^2 + 1) - 2x + 2 \operatorname{arc} \operatorname{tg} x + k \end{aligned}$$

$$e) \int (\ln x)^2 dx$$

$$\begin{cases} u = (\ln x)^2 \rightarrow du = 2(\ln x) \cdot \frac{1}{x} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int (\ln x)^2 dx = x (\ln x)^2 - 2 \int \ln x dx = x \ln^2|x| - 2x \ln|x| + 2x + k$$

$$f) \int e^{-x} \cos e^x dx = \operatorname{sen} e^x + k$$

$$g) \int \frac{1}{1-x^2} dx = \int \frac{-1}{(x+1)(x-1)} dx$$

Descomponemos en fracciones simples:

$$\frac{-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

Hallamos A y B :

$$\left. \begin{aligned} x = -1 &\rightarrow -1 = -2A \rightarrow A = 1/2 \\ x = 1 &\rightarrow -1 = 2B \rightarrow B = -1/2 \end{aligned} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{1}{1-x^2} dx &= \int \left(\frac{1/2}{x+1} + \frac{-1/2}{x-1} \right) dx = \\ &= \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + k = \ln \sqrt{\frac{x+1}{x-1}} + k \end{aligned}$$

$$\begin{aligned} h) \int \frac{(1-x)^2}{1+x} dx &= \int \frac{x^2 - 2x + 1}{x+1} dx = \int \left(x - 3 + \frac{4}{x+1} \right) dx = \\ &= \frac{x^2}{2} - 3x + 4 \ln|x+1| + k \end{aligned}$$

24 Resuelve:**S**

a) $\int \frac{1}{1+e^x} dx$

• *En el numerador, suma y resta e^x .*

b) $\int \frac{x+3}{\sqrt{9-x^2}} dx$

• *Descomponla en suma de otras dos.*

a) $\int \frac{1}{1+e^x} dx = \int \frac{1+e^x-e^x}{1+e^x} dx = \int \left(1 - \frac{e^x}{1+e^x}\right) = x - \ln(1+e^x) + k$

b) $\int \frac{x+3}{\sqrt{9-x^2}} dx = -\int \frac{-x}{\sqrt{9-x^2}} dx + \int \frac{3}{\sqrt{9-x^2}} dx =$
 $= -\sqrt{9-x^2} + 3 \int \frac{1/3}{\sqrt{1-(x/3)^2}} dx = -\sqrt{9-x^2} + 3 \operatorname{arc} \operatorname{sen}\left(\frac{x}{3}\right) + k$

25 Resuelve por sustitución:

a) $\int x\sqrt{x+1} dx$

b) $\int \frac{dx}{x-\sqrt[4]{x}}$

c) $\int \frac{x}{\sqrt{x+1}} dx$

d) $\int \frac{1}{x\sqrt{x+1}} dx$

e) $\int \frac{1}{x+\sqrt{x}} dx$

f) $\int \frac{\sqrt{x}}{1+x} dx$

• *a) Haz $x+1=t^2$. b) Haz $x=t^4$.*

a) $\int x\sqrt{x+1} dx$

Cambio: $x+1=t^2 \rightarrow dx=2t dt$

$$\int x\sqrt{x+1} dx = \int (t^2-1)t \cdot 2t dt = \int (2t^4-2t^2) dt = \frac{2t^5}{5} - \frac{2t^3}{3} + k =$$

$$= \frac{2\sqrt{(x+1)^5}}{5} - \frac{2\sqrt{(x+1)^3}}{3} + k$$

b) $\int \frac{dx}{x-\sqrt[4]{x}}$

Cambio: $x=t^4 \rightarrow dx=4t^3 dt$

$$\int \frac{dx}{x-\sqrt[4]{x}} = \int \frac{4t^3 dt}{t^4-t} = \int \frac{4t^2 dt}{t^3-1} = \frac{4}{3} \int \frac{3t^2 dt}{t^3-1} = \frac{4}{3} \ln|t^3-1| + k =$$

$$= \frac{4}{3} \ln|\sqrt[4]{x^3}-1| + k$$

$$c) \int \frac{x}{\sqrt{x+1}} dx$$

$$\text{Cambio: } x+1 = t^2 \rightarrow dx = 2t dt$$

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}} dx &= \int \frac{(t^2-1)}{t} \cdot 2t dt = \int (2t^2-2) dt = \frac{2t^3}{3} - 2t + k = \\ &= \frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + k \end{aligned}$$

$$d) \int \frac{1}{x\sqrt{x+1}} dx$$

$$\text{Cambio: } x+1 = t^2 \rightarrow dx = 2t dt$$

$$\int \frac{1}{x\sqrt{x+1}} dx = \int \frac{2t dt}{(t^2-1)t} = \int \frac{2 dt}{(t+1)(t-1)}$$

Descomponemos en fracciones simples:

$$\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}$$

$$2 = A(t-1) + B(t+1)$$

Hallamos A y B :

$$\left. \begin{aligned} t = -1 &\rightarrow 2 = -2A \rightarrow A = -1 \\ t = 1 &\rightarrow 2 = 2B \rightarrow B = 1 \end{aligned} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2 dt}{(t+1)(t-1)} &= \int \left(\frac{-1}{t+1} + \frac{1}{t-1} \right) dt = -\ln|t+1| + \ln|t-1| + k = \\ &= \ln \left| \frac{t-1}{t+1} \right| + k \end{aligned}$$

Así:

$$\int \frac{1}{x\sqrt{x+1}} dx = \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + k$$

$$e) \int \frac{1}{x+\sqrt{x}} dx$$

$$\text{Cambio: } x = t^2 \rightarrow dx = 2t dt$$

$$\begin{aligned} \int \frac{1}{x+\sqrt{x}} dx &= \int \frac{2t dt}{t^2+t} = \int \frac{2 dt}{t+1} = 2 \ln|t+1| + k = \\ &= 2 \ln(\sqrt{x}+1) + k \end{aligned}$$

$$f) \int \frac{\sqrt{x}}{1+x} dx$$

$$\text{Cambio: } x = t^2 \rightarrow dx = 2t dt$$

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{t \cdot 2t dt}{1+t^2} = \int \frac{2t^2 dt}{1+t^2} = \int \left(2 - \frac{2}{1+t^2} \right) dt = \\ &= 2t - 2 \operatorname{arc} \operatorname{tg} t + k = 2\sqrt{x} - 2 \operatorname{arc} \operatorname{tg} \sqrt{x} + k \end{aligned}$$

26 Resuelve, utilizando un cambio de variable, estas integrales:

$$a) \int \sqrt{9-4x^2} dx \quad b) \int \frac{dx}{e^{2x}-3e^x} \quad c) \int \frac{e^{3x}-e^x}{e^{2x}+1} dx \quad d) \int \frac{1}{1+\sqrt{x}} dx$$

• a) Haz $\operatorname{sen} t = 2x/3$.

$$a) \int \sqrt{9-4x^2} dx$$

$$\text{Cambio: } \operatorname{sen} t = \frac{2x}{3} \rightarrow x = \frac{3}{2} \operatorname{sen} t \rightarrow dx = \frac{3}{2} \cos t dt$$

$$\begin{aligned} \int \sqrt{9-4x^2} dx &= \int \sqrt{9-4 \cdot \frac{9}{4} \operatorname{sen}^2 t} \cdot \frac{3}{2} \cos t dt = \int 3 \cos t \cdot \frac{3}{2} \cos t dt = \\ &= \frac{9}{2} \int \cos^2 t dt = \frac{9}{2} \int \left(\frac{1}{2} - \frac{\cos 2t}{2} \right) dt = \frac{9}{2} \left(\frac{1}{2} t + \frac{1}{4} \operatorname{sen} 2t \right) + k = \\ &= \frac{9}{4} t + \frac{9}{8} \operatorname{sen} 2t + k = \frac{9}{4} \operatorname{arc} \operatorname{sen} \left(\frac{2x}{3} \right) + \frac{9}{8} \cdot 2 \operatorname{sen} t \cos t + k = \\ &= \frac{9}{4} \operatorname{arc} \operatorname{sen} \left(\frac{2x}{3} \right) + \frac{9}{4} \cdot \frac{2x}{3} \sqrt{1 - \frac{4x^2}{9}} + k = \\ &= \frac{9}{4} \operatorname{arc} \operatorname{sen} \left(\frac{2x}{3} \right) + \frac{x}{2} \cdot \sqrt{9-4x^2} + k \end{aligned}$$

$$b) \int \frac{dx}{e^{2x}-3e^x}$$

$$\text{Cambio: } e^x = t \rightarrow x = \ln t \rightarrow dx = \frac{1}{t} dt$$

$$\int \frac{dx}{e^{2x}-3e^x} = \int \frac{1/t}{t^2-3t} dt = \int \frac{1}{t^3-3t^2} dt = \int \frac{1}{t^2(t-3)} dt$$

Descomponemos en fracciones simples:

$$\frac{1}{t^2(t-3)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-3} = \frac{At(t-3) + B(t-3) + Ct^2}{t^2(t-3)}$$

$$1 = At(t-3) + B(t-3) + Ct^2$$

Hallamos A, B y C :

$$\left. \begin{aligned} t = 0 &\rightarrow 1 = -3B && \rightarrow B = -1/3 \\ t = 3 &\rightarrow 1 = 9C && \rightarrow C = 1/9 \\ t = 1 &\rightarrow 1 = -2A - 2B + C && \rightarrow A = -1/9 \end{aligned} \right\}$$

Así, tenemos que:

$$\begin{aligned} \int \frac{1}{t^2(t-3)} dt &= \int \left(\frac{-1/9}{t} + \frac{-1/3}{t^2} + \frac{1/9}{t-3} \right) dt = \\ &= \frac{-1}{9} \ln|t| + \frac{1}{3t} + \frac{1}{9} \ln|t-3| + k \end{aligned}$$

Por tanto:

$$\begin{aligned} \int \frac{dx}{e^{2x} - 3e^x} &= \frac{-1}{9} \ln e^x + \frac{1}{3e^x} + \frac{1}{9} \ln|e^x - 3| + k = \\ &= -\frac{1}{9}x + \frac{1}{3e^x} + \frac{1}{9} \ln|e^x - 3| + k \end{aligned}$$

c) $\int \frac{e^{3x} - e^x}{e^{2x} + 1} dx$

Cambio: $e^x = t \rightarrow x = \ln t \rightarrow dx = \frac{1}{t} dt$

$$\begin{aligned} \int \frac{e^{3x} - e^x}{e^{2x} + 1} dx &= \int \frac{t^3 - t}{t^2 + 1} \cdot \frac{1}{t} dt = \int \frac{t^2 - 1}{t^2 + 1} dt = \int \left(1 - \frac{2}{t^2 + 1} \right) dt = \\ &= t - 2 \operatorname{arc} \operatorname{tg} t + k = e^x - 2 \operatorname{arc} \operatorname{tg} (e^x) + k \end{aligned}$$

d) $\int \frac{1}{1 + \sqrt{x}} dx$

Cambio: $x = t^2 \rightarrow dx = 2t dt$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{x}} dx &= \int \frac{2t dt}{1 + t} = \int \left(2 - \frac{2}{1 + t} \right) dt = 2t - 2 \ln|1 + t| + k = \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + k \end{aligned}$$

27 Encuentra la primitiva de $f(x) = \frac{1}{1 + 3x}$ que se anula para $x = 0$.

S

$$F(x) = \int \frac{1}{1 + 3x} dx = \frac{1}{3} \int \frac{3}{1 + 3x} dx = \frac{1}{3} \ln|1 + 3x| + k$$

$$F(0) = k = 0$$

Por tanto: $F(x) = \frac{-1}{3} \ln|1 + 3x|$

28 Halla la función F para la que $F'(x) = \frac{1}{x^2}$ y $F(1) = 2$.

$$F(x) = \int \frac{1}{x^2} dx = \frac{-1}{x} + k$$

$$F(1) = -1 + k = 2 \Rightarrow k = 3$$

Por tanto: $F(x) = \frac{-1}{x} + 3$

29 De todas las primitivas de la función $y = 4x - 6$, ¿cuál de ellas toma el valor 4 para $x = 1$?

$$F(x) = \int (4x - 6) dx = 2x^2 - 6x + k$$

$$F(1) = 2 - 6 + k = 4 \Rightarrow k = 8$$

Por tanto: $F(x) = 2x^2 - 6x + 8$

30 Halla $f(x)$ sabiendo que $f''(x) = 6x$, $f'(0) = 1$ y $f(2) = 5$.

$$\left. \begin{array}{l} f'(x) = \int 6x dx = 3x^2 + c \\ f'(0) = c = 1 \end{array} \right\} f'(x) = 3x^2 + 1$$

$$\left. \begin{array}{l} f(x) = \int (3x^2 + 1) dx = x^3 + x + k \\ f(2) = 10 + k = 5 \Rightarrow k = -5 \end{array} \right\}$$

Por tanto: $f(x) = x^3 + x - 5$

31 Resuelve las siguientes integrales por sustitución:

a) $\int \frac{e^x}{1 - \sqrt{e^x}} dx$

b) $\int \sqrt{e^x - 1} dx$

• a) Haz $\sqrt{e^x} = t$. b) Haz $\sqrt{e^x - 1} = t$.

a) $\int \frac{e^x}{1 - \sqrt{e^x}} dx$

Cambio: $\sqrt{e^x} = t \rightarrow e^{x/2} = t \rightarrow \frac{x}{2} = \ln t \rightarrow dx = \frac{2}{t} dt$

$$\int \frac{e^x}{1 - \sqrt{e^x}} = \int \frac{t^2 \cdot (2/t) dt}{1 - t} = \int \frac{2t dt}{1 - t} = \int \left(-2 + \frac{2}{1 - t} \right) dt =$$

$$= -2t - 2 \ln |1 - t| + k = -2\sqrt{e^x} - 2 \ln |1 - \sqrt{e^x}| + k$$

$$b) \int \sqrt{e^x - 1} \, dx$$

$$\text{Cambio: } \sqrt{e^x - 1} = t \rightarrow e^x = t^2 + 1 \rightarrow x = \ln(t^2 + 1) \rightarrow dx = \frac{2t}{t^2 + 1} dt$$

$$\begin{aligned} \int \sqrt{e^x - 1} \, dx &= \int t \cdot \frac{2t}{t^2 + 1} dt = \int \frac{2t^2}{t^2 + 1} dt = \int \left(2 - \frac{2}{t^2 + 1} \right) dt = \\ &= 2t - 2 \operatorname{arc} \operatorname{tg} t + k = 2\sqrt{e^x - 1} - 2 \operatorname{arc} \operatorname{tg} \sqrt{e^x - 1} + k \end{aligned}$$

32 Calcula $\int \frac{\operatorname{sen}^2 x}{1 + \cos x} dx$.

• Multiplica numerador y denominador por $1 - \cos x$.

$$\begin{aligned} \int \frac{\operatorname{sen}^2 x}{1 + \cos x} dx &= \int \frac{\operatorname{sen}^2 x (1 - \cos x)}{(1 + \cos x)(1 - \cos x)} dx = \int \frac{\operatorname{sen}^2 x (1 - \cos x)}{1 - \cos^2 x} dx = \\ &= \int \frac{\operatorname{sen}^2 x (1 - \cos x)}{\operatorname{sen}^2 x} dx = \int (1 - \cos x) dx = x - \operatorname{sen} x + k \end{aligned}$$

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33 Encuentra una primitiva de la función:

S

$$f(x) = x^2 \operatorname{sen} x$$

cuyo valor para $x = \pi$ sea 4.

$$F(x) = \int x^2 \operatorname{sen} x \, dx$$

Integramos por partes:

$$\begin{cases} u = x^2 \rightarrow du = 2x \, dx \\ dv = \operatorname{sen} x \, dx \rightarrow v = -\cos x \end{cases}$$

$$F(x) = -x^2 \cos x + 2 \underbrace{\int x \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x \rightarrow du_1 = dx \\ dv_1 = \cos x \, dx \rightarrow v_1 = \operatorname{sen} x \end{cases}$$

$$I_1 = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x + \cos x$$

Por tanto:

$$\left. \begin{aligned} F(x) &= -x^2 \cos x + 2x \operatorname{sen} x + 2 \cos x + k \\ F(\pi) &= \pi^2 - 2 + k = 4 \Rightarrow k = 6 - \pi^2 \end{aligned} \right\}$$

$$F(x) = -x^2 \cos x + 2x \operatorname{sen} x + 2 \cos x + 6 - \pi^2$$

34 Determina la función $f(x)$ sabiendo que:

S

$$f''(x) = x \ln x, \quad f'(1) = 0 \quad \text{y} \quad f(e) = \frac{e}{4}$$

$$f'(x) = \int x \ln x \, dx$$

Integramos por partes:

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = x \, dx \rightarrow v = \frac{x^2}{2} \end{cases}$$

$$\left. \begin{aligned} f'(x) &= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + k = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + k \\ f'(1) &= \frac{1}{2} \left(-\frac{1}{2} \right) + k = -\frac{1}{4} + k = 0 \Rightarrow k = \frac{1}{4} \end{aligned} \right\}$$

$$f'(x) = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + \frac{1}{4}$$

$$f(x) = \int \left[\frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + \frac{1}{4} \right] dx = \underbrace{\int \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) dx}_I + \frac{1}{4} x$$

$$\begin{cases} u = \left(\ln x - \frac{1}{2} \right) \rightarrow du = \frac{1}{x} dx \\ dv = \frac{x^2}{2} dx \rightarrow v = \frac{x^3}{6} \end{cases}$$

$$I = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \int \frac{x^2}{6} \, dx = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + k$$

Por tanto:

$$\left. \begin{aligned} f(x) &= \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + \frac{1}{4} x + k \\ f(e) &= \frac{e^3}{12} - \frac{e^3}{18} + \frac{e}{4} + k = \frac{e^3}{36} + \frac{e}{4} + k = \frac{e}{4} \Rightarrow k = -\frac{e^3}{36} \end{aligned} \right\}$$

$$f(x) = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + \frac{1}{4} x - \frac{e^3}{36}$$

- 35** **S** Calcula la expresión de una función $f(x)$ tal que $f'(x) = x e^{-x^2}$ y que $f(0) = \frac{1}{2}$.

$$f(x) = \int x e^{-x^2} dx = -\frac{1}{2} \int -2x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + k$$

$$f(0) = -\frac{1}{2} + k = \frac{1}{2} \Rightarrow k = 1$$

Por tanto: $f(x) = -\frac{1}{2} e^{-x^2} + 1$

- 36** **S** Encuentra la función derivable $f: [-1, 1] \rightarrow \mathbb{R}$ que cumple $f(1) = -1$ y

$$f'(x) = \begin{cases} x^2 - 2x & \text{si } -1 \leq x < 0 \\ e^x - 1 & \text{si } 0 \leq x \leq 1 \end{cases}$$

- Si $x \neq 0$:

$$f(x) = \begin{cases} \frac{x^3}{3} - x^2 + k & \text{si } -1 \leq x < 0 \\ e^x - x + c & \text{si } 0 < x \leq 1 \end{cases}$$

- Hallamos k y c teniendo en cuenta que $f(1) = -1$ y que $f(x)$ ha de ser continua en $x = 0$.

$$f(1) = -1 \Rightarrow e - 1 + c = -1 \Rightarrow c = -e$$

$$\left. \begin{array}{l} \lim_{x \rightarrow 0^-} f(x) = k \\ \lim_{x \rightarrow 0^+} f(x) = 1 - e \end{array} \right\} k = 1 - e$$

$$\text{Por tanto: } f(x) = \begin{cases} \frac{x^3}{3} - x^2 + 1 - e & \text{si } -1 \leq x < 0 \\ e^x - x - e & \text{si } 0 \leq x \leq 1 \end{cases}$$

- 37** **S** De una función derivable se sabe que pasa por el punto $A(-1, -4)$ y que su derivada es:

$$f'(x) = \begin{cases} 2 - x & \text{si } x \leq 1 \\ 1/x & \text{si } x > 1 \end{cases}$$

a) Halla la expresión de $f(x)$.

b) Obtén la ecuación de la recta tangente a $f(x)$ en $x = 2$.

a) Si $x \neq 1$:

$$f(x) = \begin{cases} 2x - \frac{x^2}{2} + k & \text{si } x < 1 \\ \ln x + c & \text{si } x > 1 \end{cases}$$

Hallamos k y c teniendo en cuenta que $f(-1) = -4$ y que $f(x)$ ha de ser continua en $x = 1$.

$$f(-1) = -\frac{5}{2} + k = -4 \Rightarrow k = -\frac{3}{2}$$

$$\left. \begin{array}{l} \lim_{x \rightarrow 1^-} f(x) = \frac{3}{2} - \frac{3}{2} = 0 \\ \lim_{x \rightarrow 1^+} f(x) = c \end{array} \right\} c = 0$$

$$\text{Por tanto: } f(x) = \begin{cases} 2x - \frac{x^2}{2} - \frac{3}{2} & \text{si } x < 1 \\ \ln x & \text{si } x \geq 1 \end{cases}$$

$$\text{b) } f(2) = \ln 2; \quad f'(2) = \frac{1}{2}$$

La ecuación de la recta tangente será: $y = \ln 2 + \frac{1}{2}(x - 2)$

38 **Calcula:**

S

$$\text{a) } \int |1 - x| dx \quad \text{b) } \int (3 + |x|) dx \quad \text{c) } \int |2x - 1| dx \quad \text{d) } \int \left| \frac{x}{2} - 2 \right| dx$$

$$\text{a) } \int |1 - x| dx$$

$$|1 - x| = \begin{cases} 1 - x & \text{si } x < 1 \\ -1 + x & \text{si } x \geq 1 \end{cases}$$

$$f(x) = \int |1 - x| dx = \begin{cases} x - \frac{x^2}{2} + k & \text{si } x < 1 \\ -x + \frac{x^2}{2} + c & \text{si } x \geq 1 \end{cases}$$

En $x = 1$, la función ha de ser continua:

$$\left. \begin{array}{l} \lim_{x \rightarrow 1^-} f(x) = \frac{1}{2} + k \\ \lim_{x \rightarrow 1^+} f(x) = -\frac{1}{2} + c \end{array} \right\} \frac{1}{2} + k = -\frac{1}{2} + c \Rightarrow c = 1 + k$$

Por tanto:

$$\int |1 - x| dx = \begin{cases} x - \frac{x^2}{2} + k & \text{si } x < 1 \\ -x + \frac{x^2}{2} + 1 + k & \text{si } x \geq 1 \end{cases}$$

$$b) \int (3 + |x|) dx$$

$$3 + |x| = \begin{cases} 3 - x & \text{si } x < 0 \\ 3 + x & \text{si } x \geq 0 \end{cases}$$

$$f(x) = \int (3 + |x|) dx = \begin{cases} 3x - \frac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \frac{x^2}{2} + c & \text{si } x \geq 0 \end{cases}$$

En $x = 0$, $f(x)$ ha de ser continua:

$$\left. \begin{array}{l} \lim_{x \rightarrow 0^-} f(x) = k \\ \lim_{x \rightarrow 0^+} f(x) = c \end{array} \right\} c = k$$

Por tanto:

$$\int (3 + |x|) dx = \begin{cases} 3x - \frac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \frac{x^2}{2} + k & \text{si } x \geq 0 \end{cases}$$

$$c) \int |2x - 1| dx$$

$$|2x - 1| = \begin{cases} -2x + 1 & \text{si } x < 1/2 \\ 2x - 1 & \text{si } x \geq 1/2 \end{cases}$$

$$f(x) = \int |2x - 1| dx = \begin{cases} -x^2 + x + k & \text{si } x < \frac{1}{2} \\ x^2 - x + c & \text{si } x \geq \frac{1}{2} \end{cases}$$

$f(x)$ ha de ser continua en $x = \frac{1}{2}$:

$$\left. \begin{array}{l} \lim_{x \rightarrow (1/2)^-} f(x) = \frac{1}{4} + k \\ \lim_{x \rightarrow (1/2)^+} f(x) = -\frac{1}{4} + c \end{array} \right\} \frac{1}{4} + k = -\frac{1}{4} + c \Rightarrow c = \frac{1}{2} + k$$

Por tanto:

$$\int |2x - 1| dx = \begin{cases} -x^2 + x + k & \text{si } x < \frac{1}{2} \\ x^2 - x + \frac{1}{2} + k & \text{si } x \geq \frac{1}{2} \end{cases}$$

$$d) \int \left| \frac{x}{2} - 2 \right| dx$$

$$\left| \frac{x}{2} - 2 \right| = \begin{cases} -\frac{x}{2} + 2 & \text{si } x < 4 \\ \frac{x}{2} - 2 & \text{si } x \geq 4 \end{cases}$$

$$f(x) = \int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + c & \text{si } x \geq 4 \end{cases}$$

$f(x)$ ha de ser continua en $x = 4$:

$$\left. \begin{array}{l} \lim_{x \rightarrow 4^-} f(x) = 4 + k \\ \lim_{x \rightarrow 4^+} f(x) = -4 + c \end{array} \right\} 4 + k = -4 + c \Rightarrow c = 8 + k$$

Por tanto:

$$\int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + 8 + k & \text{si } x \geq 4 \end{cases}$$

39 Calcula $\int \frac{1}{\operatorname{sen}^2 x \cos^2 x} dx$.

$$\begin{aligned} \int \frac{1}{\operatorname{sen}^2 x \cos^2 x} dx &= \int \frac{\operatorname{sen}^2 x + \cos^2 x}{\operatorname{sen}^2 x \cos^2 x} dx = \\ &= \int \frac{\operatorname{sen}^2 x}{\operatorname{sen}^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\operatorname{sen}^2 x \cos^2 x} dx = \\ &= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\operatorname{sen}^2 x} dx = \operatorname{tg} x - \operatorname{cotg} x + k \end{aligned}$$

CUESTIONES TEÓRICAS

40 Prueba que, si $F(x)$ es una primitiva de $f(x)$ y C un número real cualquiera, la función $F(x) + C$ es también una primitiva de $f(x)$.

$$F(x) \text{ primitiva de } f(x) \Leftrightarrow F'(x) = f(x)$$

$$(F(x) + C)' = F'(x) = f(x) \Rightarrow F(x) + C \text{ es primitiva de } f(x).$$

- 41 Representa tres primitivas de la función f cuya gráfica es esta:**

$$f(x) = 2 \Rightarrow F(x) = 2x + k$$

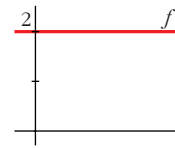
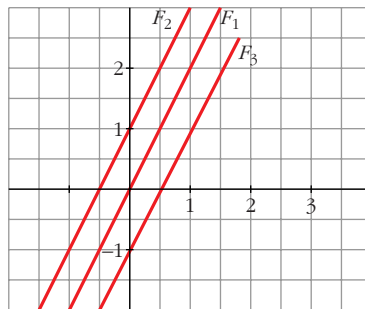
Por ejemplo:

$$F_1(x) = 2x$$

$$F_2(x) = 2x + 1$$

$$F_3(x) = 2x - 1$$

cuyas gráficas son:



- 42 Representa tres primitivas de la función f :**

$$f(x) = 2x \Rightarrow F(x) = x^2 + k$$

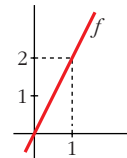
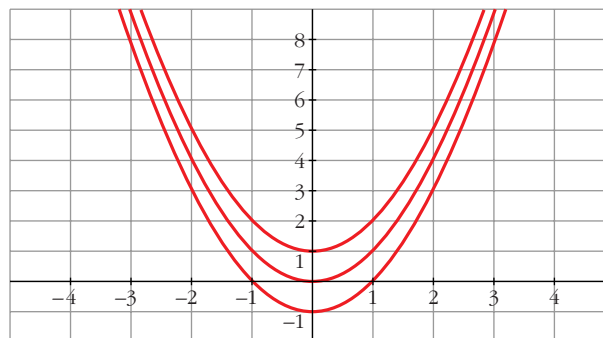
Por ejemplo:

$$F_1(x) = x^2$$

$$F_2(x) = x^2 + 1$$

$$F_3(x) = x^2 - 1$$

cuyas gráficas son:



- 43 Sabes que una primitiva de la función $f(x) = \frac{1}{x}$ es $F(x) = \ln |x|$. ¿Por qué se toma el valor absoluto de x ?**

$f(x) = \frac{1}{x}$ está definida para todo $x \neq 0$; y es la derivada de la función:

$$F(x) = \begin{cases} \ln x & \text{si } x > 0 \\ \ln (-x) & \text{si } x < 0 \end{cases}$$

es decir, de $F(x) = \ln |x|$.

- 44 En una integral hacemos el cambio de variable $e^x = t$. ¿Cuál es la expresión de dx en función de t ?**

$$e^x = t \rightarrow x = \ln t \rightarrow dx = \frac{1}{t} dt$$

45 Comprueba que: $\int \frac{1}{\cos x} dx = \ln |\sec x + \operatorname{tg} x| + k$

Tenemos que probar que la derivada de $f(x) = \ln |\sec x + \operatorname{tg} x| + k$ es $f'(x) = \frac{1}{\cos x}$.

Derivamos $f(x) = \ln \left| \frac{1 + \operatorname{sen} x}{\cos x} \right| + k$:

$$\begin{aligned} f'(x) &= \frac{\frac{\cos^2 x + \operatorname{sen} x(1 + \operatorname{sen} x)}{\cos^2 x}}{\frac{1 + \operatorname{sen} x}{\cos x}} = \frac{\frac{\cos^2 x + \operatorname{sen} x + \operatorname{sen}^2 x}{\cos^2 x}}{1 + \operatorname{sen} x} = \\ &= \frac{1 + \operatorname{sen} x}{(1 + \operatorname{sen} x) \cos x} = \frac{1}{\cos x} \end{aligned}$$

46 Comprueba que: $\int \frac{1}{\operatorname{sen} x \cos x} dx = \ln |\operatorname{tg} x| + k$

Tenemos que comprobar que la derivada de la función $f(x) = \ln |\operatorname{tg} x| + k$ es

$$f'(x) = \frac{1}{\operatorname{sen} x \cos x}.$$

Derivamos $f(x)$:

$$f'(x) = \frac{1/\cos^2 x}{\operatorname{tg} x} = \frac{1/\cos^2 x}{\operatorname{sen} x/\cos x} = \frac{1}{\operatorname{sen} x \cos x}$$

47 Sin utilizar cálculo de derivadas, prueba que:

$$F(x) = \frac{1}{1+x^4} \quad \text{y} \quad G(x) = \frac{-x^4}{1+x^4}$$

son dos primitivas de una misma función.

Si $F(x)$ y $G(x)$ son dos primitivas de una misma función, su diferencia es una constante. Veámoslo:

$$F(x) - G(x) = \frac{1}{1+x^4} - \left(\frac{-x^4}{1+x^4} \right) = \frac{1+x^4}{1+x^4} = 1$$

Por tanto, hemos obtenido que: $F(x) = G(x) + 1$

Luego las dos son primitivas de una misma función.

48 Sean f y g dos funciones continuas y derivables que se diferencian en una constante. ¿Podemos asegurar que f y g tienen una misma primitiva?

No. Por ejemplo:

$$\left. \begin{aligned} f(x) = 2x + 1 &\rightarrow F(x) = x^2 + x + k \\ g(x) = 2x + 2 &\rightarrow G(x) = x^2 + 2x + c \end{aligned} \right\}$$

$f(x)$ y $g(x)$ son continuas, derivables y se diferencian en una constante (pues $f(x) = g(x) - 1$).

Sin embargo, sus primitivas, $F(x)$ y $G(x)$ respectivamente, son distintas, cualesquiera que sean los valores de k y c .

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PARA PROFUNDIZAR

49 Para integrar una función cuyo denominador es un polinomio de segundo grado sin raíces reales, distinguiremos dos casos:

a) Si el numerador es constante, transformamos el denominador para obtener un binomio al cuadrado. La solución será un arco tangente:

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x + 2)^2 + 1}$$

(Completa la resolución).

b) Si el numerador es de primer grado, se descompone en un logaritmo neperiano y un arco tangente:

$$\int \frac{(x + 5) dx}{x^2 + 2x + 3} = \frac{1}{2} \int \frac{2x + 10}{x^2 + 2x + 3} dx = \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 3} dx + \frac{1}{2} \int \frac{8 dx}{x^2 + 2x + 3}$$

(Completa su resolución).

$$a) \int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x + 2)^2 + 1} = \text{arc tg}(x + 2) + k$$

$$b) \int \frac{(x + 5) dx}{x^2 + 2x + 3} = \frac{1}{2} \int \frac{2x + 10}{x^2 + 2x + 3} dx = \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 3} dx + \frac{1}{2} \int \frac{8 dx}{x^2 + 2x + 3} =$$

$$= \frac{1}{2} \ln(x^2 + 2x + 3) + 4 \int \frac{dx}{(x + 1)^2 + 2} =$$

$$= \frac{1}{2} \ln(x^2 + 2x + 3) + 2 \int \frac{dx}{\left(\frac{x + 1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2} \ln(x^2 + 2x + 3) + 2\sqrt{2} \int \frac{(1/\sqrt{2}) dx}{\left(\frac{x + 1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2} \ln(x^2 + 2x + 3) + 2\sqrt{2} \text{ arc tg} \left(\frac{x + 1}{\sqrt{2}} \right) + k$$

50 Observa cómo se resuelve esta integral:

$$I = \int \frac{x+1}{x^3 + 2x^2 + 3x} dx$$

$$x^3 + 2x^2 + 3x = x(x^2 + 2x + 3)$$

La fracción se descompone así: $\frac{x+1}{x^3 + 2x^2 + 3x} = \frac{A}{x} + \frac{Bx+C}{x^2 + 2x + 3}$

Obtenemos: $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = \frac{1}{3}$

Sustituimos: $I = \frac{1}{3} \int \frac{1}{x} dx - \frac{1}{3} \int \frac{x-1}{x^2 + 2x + 3} dx$

(Completa su resolución).

Completamos la resolución:

$$\begin{aligned} I &= \frac{1}{3} \int \frac{1}{x} dx - \frac{1}{3} \int \frac{x-1}{x^2 + 2x + 3} dx = \\ &= \frac{1}{3} \ln|x| - \frac{1}{6} \int \frac{2x-2}{x^2 + 2x + 3} dx = \frac{1}{3} \ln|x| - \frac{1}{6} \int \frac{2x+2-4}{x^2 + 2x + 3} dx = \\ &= \frac{1}{3} \ln|x| - \frac{1}{6} \int \frac{2x-2}{x^2 + 2x + 3} dx + \frac{2}{3} \int \frac{dx}{x^2 + 2x + 3} \quad (*) \\ &= \frac{1}{3} \ln|x| - \frac{1}{6} \ln(x^2 + 2x + 3) + \frac{\sqrt{2}}{3} \operatorname{arc\,tg} \left(\frac{x+1}{\sqrt{2}} \right) + k \end{aligned}$$

(*) (Ver en el ejercicio 49 apartado b) el cálculo de $\int \frac{dx}{x^2 + 2x + 3}$).

51 Resuelve las siguientes integrales:

a) $\int \frac{2x-1}{x^3+x} dx$ **b)** $\int \frac{1}{x^3+1} dx$ c) $\int \frac{x^2+3x+8}{x^2+9} dx$

d) $\int \frac{2x+10}{x^2+x+1} dx$ e) $\int \frac{2}{x^2+3x+4} dx$ f) $\int \frac{dx}{(x+1)^2(x^2+1)}$

• e) *Multiplica numerador y denominador por 4.*

a) $\int \frac{2x-1}{x^3+x} dx = \int \frac{2x-1}{x(x^2+1)} dx$

Descomponemos la fracción:

$$\frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + Bx^2 + Cx}{x(x^2+1)}$$

$$2x-1 = A(x^2+1) + Bx^2 + Cx$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = 0 \rightarrow -1 = A \\ x = 1 \rightarrow 1 = 2A + B + C \rightarrow 3 = B + C \\ x = -1 \rightarrow -3 = 2A + B - C \rightarrow -1 = B - C \end{array} \right\} \begin{array}{l} A = -1 \\ B = 1 \\ C = 2 \end{array}$$

Por tanto:

$$\begin{aligned} \int \frac{2x-1}{x^3+x} dx &= \int \left(\frac{-1}{x} + \frac{x+2}{x^2+1} \right) dx = \\ &= \int \frac{-1}{x} dx + \frac{1}{2} \int \frac{2x}{x^2+1} dx + 2 \int \frac{dx}{x^2+1} = \\ &= -\ln|x| + \frac{1}{2} \ln(x^2+1) + 2 \operatorname{arc} \operatorname{tg} x + k \end{aligned}$$

$$\text{b) } \int \frac{1}{x^3+1} dx = \int \frac{dx}{(x+1)(x^2-x+1)}$$

Descomponemos la fracción:

$$\begin{aligned} \frac{1}{(x+1)(x^2-x+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \\ &= \frac{A(x^2-x+1) + Bx(x+1) + C(x+1)}{(x+1)(x^2-x+1)} \end{aligned}$$

$$1 = A(x^2-x+1) + Bx(x+1) + C(x+1)$$

Hallamos A , B y C :

$$\left. \begin{array}{l} x = -1 \rightarrow 1 = 3A \rightarrow A = 1/3 \\ x = 0 \rightarrow 1 = A + C \rightarrow C = 2/3 \\ x = 1 \rightarrow 1 = A + 2B + 2C \rightarrow B = -1/3 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{1}{x^3+1} dx &= \int \frac{-1/3}{x+1} dx + \int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1} dx = \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{3} \int \frac{x-2}{x^2-x+1} dx = \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \int \frac{2x-4}{x^2-x+1} dx = \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \int \frac{2x-1-3}{x^2-x+1} dx = \\ &= \frac{1}{3} \ln|x+1| - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{dx}{x^2-x+1} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} = \\
&= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{1}{2} \int \frac{4/3}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx = \\
&= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{\sqrt{3}}{3} \int \frac{2/\sqrt{3}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx = \\
&= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) + \frac{\sqrt{3}}{3} \operatorname{arc\,tg}\left(\frac{2x-1}{\sqrt{3}}\right) + k
\end{aligned}$$

$$\begin{aligned}
\text{c) } \int \frac{x^2 + 3x + 8}{x^2 + 9} dx &= \int \left(1 + \frac{3x-1}{x^2+9}\right) dx = x + \int \frac{3x}{x^2+9} dx - \int \frac{dx}{x^2+9} = \\
&= x + \frac{3}{2} \int \frac{2x}{x^2+9} dx - \int \frac{1/9}{(x/3)^2+1} dx = \\
&= x + \frac{3}{2} \ln(x^2+9) - \frac{1}{3} \operatorname{arc\,tg}\left(\frac{x}{3}\right) + k
\end{aligned}$$

$$\begin{aligned}
\text{d) } \int \frac{2x+10}{x^2+x+1} dx &= \int \frac{2x+1+9}{x^2+x+1} dx = \int \frac{2x+1}{x^2+x+1} dx + 9 \int \frac{1}{x^2+x+1} dx = \\
&= \ln(x^2+x+1) + 9 \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} = \\
&= \ln(x^2+x+1) + 6\sqrt{3} \int \frac{2/\sqrt{3}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx = \\
&= \ln(x^2+x+1) + 6\sqrt{3} \operatorname{arc\,tg}\left(\frac{2x+1}{\sqrt{3}}\right) + k
\end{aligned}$$

$$\begin{aligned}
\text{e) } \int \frac{2}{x^2+3x+4} dx &= \int \frac{8}{4x^2+12x+16} dx = \int \frac{8}{(2x+3)^2+7} dx = \\
&= \int \frac{8/7}{\left(\frac{2x+3}{\sqrt{7}}\right)^2+1} dx = \frac{8}{7} \cdot \frac{\sqrt{7}}{2} \int \frac{2/\sqrt{7}}{\left(\frac{2x+3}{\sqrt{7}}\right)^2+1} dx = \\
&= \frac{4\sqrt{7}}{7} \operatorname{arc\,tg}\left(\frac{2x+3}{\sqrt{7}}\right) + k
\end{aligned}$$

$$f) \int \frac{dx}{(x+1)^2(x^2+1)}$$

Descomponemos la fracción:

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x^2+1) + Cx(x+1)^2 + D(x+1)^2$$

Hallamos A , B , C y D :

$$\left. \begin{array}{l} x = -1 \rightarrow 1 = 2B \rightarrow B = 1/2 \\ x = 0 \rightarrow 1 = A + B + D \\ x = 1 \rightarrow 1 = 4A + 2B + 4C + 4D \\ x = -2 \rightarrow 1 = -5A + 5B - 2C + D \end{array} \right\} \begin{array}{l} A = 1/2 \\ B = 1/2 \\ C = -1/2 \\ D = 0 \end{array}$$

Por tanto:

$$\begin{aligned} \int \frac{dx}{(x+1)^2(x^2+1)} &= \int \left(\frac{1/2}{x+1} + \frac{1/2}{(x+1)^2} - \frac{1}{2} \cdot \frac{x}{x^2+1} \right) dx = \\ &= \frac{1}{2} \ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4} \ln(x^2+1) + k \end{aligned}$$

PARA PENSAR UN POCO MÁS

52 Se llama ecuación diferencial de primer orden a una ecuación en la que, además de las variables x e y , figura también y' . Resolver una ecuación diferencial es buscar una función $y = f(x)$ que verifique la ecuación propuesta.

Por ejemplo, la ecuación $x y^2 + y' = 0$ se resuelve así:

$$y' = -x y^2 \rightarrow \frac{dy}{dx} = -x y^2 \rightarrow dy = -x y^2 dx$$

Separamos las variables:

$$\frac{dy}{y^2} = -x dx \rightarrow \int \frac{dy}{y^2} = \int (-x) dx$$

$$-\frac{1}{y} = -\frac{x^2}{2} + k \rightarrow y = \frac{2}{x^2 - 2k}$$

Hay infinitas soluciones.

Busca la solución que pasa por el punto $(0, 2)$ y comprueba que la curva que obtienes verifica la ecuación propuesta.

- Buscamos la solución que pasa por el punto $(0, 2)$:

$$y = \frac{2}{x^2 - 2k} \rightarrow 2 = \frac{2}{-2k} \Rightarrow -4k = 2 \Rightarrow k = \frac{-1}{2}$$

Por tanto: $y = \frac{2}{x^2 + 1}$

• Comprobamos que verifica la ecuación $xy^2 + y' = 0$:

$$\begin{aligned} xy^2 + y' &= x \left(\frac{2}{x^2 + 1} \right)^2 - \frac{4x}{(x^2 + 1)^2} = x \cdot \frac{4}{(x^2 + 1)^2} - \frac{4x}{(x^2 + 1)^2} = \\ &= \frac{4x}{(x^2 + 1)^2} - \frac{4x}{(x^2 + 1)^2} = 0 \end{aligned}$$

53 Resuelve las siguientes ecuaciones:

a) $yy' - x = 0$

b) $y^2 y' - x^2 = 1$

c) $y' - xy = 0$

d) $y' \sqrt{x} - y = 0$

e) $y' e^y + 1 = e^x$

f) $x^2 y' + y^2 + 1 = 0$

a) $yy' - x = 0$

$$\begin{aligned} y' &= \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow y dy = x dx \Rightarrow \int y dy = \int x dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + k \Rightarrow y^2 = x^2 + 2k \end{aligned}$$

b) $y^2 y' - x^2 = 1$

$$\begin{aligned} y' &= \frac{1 + x^2}{y^2} \Rightarrow \frac{dy}{dx} = \frac{1 + x^2}{y^2} \Rightarrow y^2 dy = (1 + x^2) dx \\ \int y^2 dy &= \int (1 + x^2) dx \Rightarrow \frac{y^3}{3} = x + \frac{x^3}{3} + k \Rightarrow \\ &\Rightarrow y^3 = 3x + x^3 + 3k \Rightarrow y = \sqrt[3]{3x + x^3 + 3k} \end{aligned}$$

c) $y' - xy = 0$

$$\begin{aligned} y' &= xy \Rightarrow \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x dx \Rightarrow \int \frac{dy}{y} = \int x dx \\ \ln |y| &= \frac{x^2}{2} + k \Rightarrow |y| = e^{(x^2/2) + k} \end{aligned}$$

d) $y' \sqrt{x} - y = 0$

$$\begin{aligned} y' &= \frac{y}{\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{y}{\sqrt{x}} \Rightarrow \frac{dy}{y} = \frac{dx}{\sqrt{x}} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{\sqrt{x}} \\ \ln |y| &= 2\sqrt{x} + k \Rightarrow |y| = e^{2\sqrt{x} + k} \end{aligned}$$

e) $y' e^y + 1 = e^x$

$$\begin{aligned} y' &= \frac{e^x - 1}{e^y} \Rightarrow \frac{dy}{dx} = \frac{e^x - 1}{e^y} \\ e^y dy &= (e^x - 1) dx \Rightarrow \int e^y dy = \int (e^x - 1) dx \\ e^y &= e^x - x + k \Rightarrow y = \ln(e^x - x + k) \end{aligned}$$

$$f) x^2 y' + y^2 + 1 = 0$$

$$y' = \frac{-1 - y^2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{-(1 + y^2)}{x^2} \Rightarrow \frac{dy}{1 + y^2} = \frac{-1}{x^2} dx$$

$$\int \frac{dy}{1 + y^2} = \int \frac{-1}{x^2} dx \Rightarrow \text{arc tg } y = \frac{1}{x} + k$$

$$y = \text{tg} \left(\frac{1}{x} + k \right)$$